A ghost ring for the left-free double Burnside ring and an application to fusion systems[*]

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Abstract

For a finite group G, we define a ghost ring and a mark homomorphism for the double Burnside ring $B^{\triangleleft}(G, G)$ of left-free (G, G)-bisets. In analogy to the case of the Burnside ring $B(G)$, the ghost ring has a much simpler ring structure, and after tensoring with $\mathbb{Q}$ one obtains an isomorphism of $\mathbb{Q}$-algebras. As an application of a key lemma, we obtain a very general formula for the Brauer construction applied to a tensor product of two p-permutation bimodules M and N in terms of Brauer constructions of the bimodules M and N. Over a field of characteristic 0 we determine the simple modules of the left-free double Burnside algebra and prove semisimplicity results for the bifree double Burnside algebra. These results carry over to results about biset-functor categories. Finally, we apply the ghost ring and mark homomorphism to fusion systems on a finite p-group. We extend a remarkable bijection, due to Ragnarsson and Stancu, between saturated fusion systems and certain idempotents of the bifree double Burnside algebra over $\mathbb{Z}_{(p)}$ to a bijection between all fusion systems and a larger set of idempotents in the bifree double Burnside algebra over $\mathbb{Q}$.

Introduction

The *Burnside ring* $B(G)$ of a finite group G, the Grothendieck ring of the category of finite G-sets, has proved to be an object of central importance for the representation theory of finite groups. It is an initial object in functorial approaches to the representation theory of finite groups, in the sense that there is a unique functorial homomorphism from the Burnside ring to any other representation ring of G, for example to the character ring. Induction theorems that are now cornerstones of representation theory are proved through the study of these homomorphisms, see [D1]. The Burnside ring is also an important invariant of the group G itself. For instance, a fundamental theorem of Dress, cf. [D2], states

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that G is soluble if and only if $B(G)$ is connected, i.e., if 0 and 1 are the only idempotents of $B(G)$. The main tool in achieving these results and studying the ring structure of $B(G)$ is the *mark homomorphism* $\Phi\colon B(G) \to \prod_{H \leqslant G} \mathbb{Z}$; it is an injective ring homomorphism whose image is contained in the ring $\tilde{B}(G)$ of G-fixed points of the latter product ring, where G acts by permutation of the components with respect to the conjugation action on the set of subgroups H of G. The ring $\tilde{B}(G)$ is often called the *ghost ring* of $B(G)$. It can be interpreted as the integral closure of $B(G)$ in $\mathbb{Q}B(G) := \mathbb{Q} \otimes_{\mathbb{Z}} B(G)$. The image of Φ has finite index in $\tilde{B}(G)$, so that Φ induces an isomorphism of $\mathbb{Q}$-algebras $\mathbb{Q}B(G) \to (\prod_{H \leqslant G} \mathbb{Q})^G = \mathbb{Q}\tilde{B}(G)$ after scalar extension to $\mathbb{Q}$.

The main purpose of this paper is to construct a ghost ring and a mark homomorphism for certain subrings of the *double Burnside ring* $B(G,G)$ such that these ghost rings have a simpler multiplicative structure. The ring $B(G,G)$ is the Grothendieck group of the category of finite (G,G)-bisets. More generally, for finite groups G and H, one constructs $B(G,H)$ as the Grothendieck group of the category of finite (G,H)-bisets. If K is another finite group, a construction on bisets that is similar to the tensor product of bimodules induces a bilinear map $B(G,H) \times B(H,K) \to B(G,K)$. This defines the ring structure on $B(G,G)$. Considering the subcategories of left-free and bifree (G,H)-bisets, one obtains subgroups $B^\Delta(G,H) \subseteq B^\lhd(G,H) \subseteq B(G,H)$ that are stable under the bilinear map. In particular, one has unitary subrings $B^\Delta(G,G) \subseteq B^\lhd(G,G) \subseteq B(G,G)$. One should point out that these rings are not commutative, in contrast to $B(G)$. We refer the reader to work of Bouc, cf. [Bc96a] and [Bc10], for motivations and fundamentals on the double Burnside group. More recently, this group has become a focus of attention through applications to modular representation theory (cf. [Bc06]), connections with algebraic topology (cf. [MP], [BLO03]) and connections with fusion systems on p-groups (cf. [BLO04], [R] and [RS]).

In this paper we construct bifree and left-free ghost groups $\tilde{B}^\Delta(G,H) \subseteq \tilde{B}^\lhd(G,H)$ and a mark homomorphism $\rho_{G,H}\colon B^\lhd(G,H) \to \tilde{B}^\lhd(G,H)$, which restricts to a homomorphism $B^\Delta(G,H) \to \tilde{B}^\Delta(G,H)$. We also construct a bilinear map $\tilde{B}^\lhd(G,H) \times \tilde{B}^\lhd(H,K) \to \tilde{B}^\lhd(G,K)$ such that the maps ρ commute with the bilinear maps. We show that $\rho_{G,H}$ is injective and has finite cokernel. The construction of the ghost ring and the mark homomorphism is carried out in Section 4, and the main properties are proved in Theorem 4.7.

The key observation to the construction of the ghost rings and mark homomorphisms is Theorem 2.3, which shows how to express fixed points of the tensor product of two left-free bisets X and Y in terms of fixed points of X and Y. As an immediate application we derive a formula (see Theorem 3.2) for the Brauer construction of the tensor product of two p-permutation bimodules M and N in terms of the Brauer constructions of M and N. Special cases and variations of this formula were known before, see for instance [Ri, Section 4], [BX, Corollary 3.6] and [L2, Theorem 9.2].

One obvious application of the ghost rings and mark homomorphisms is that they give a different perspective on the ring structures of $B^\Delta(G,G)$ and $B^\lhd(G,G)$, especially if tensored with $\mathbb{Q}$. The ghost ring of $B^\Delta(G,G)$ decomposes into a direct product of rings, indexed by the isomorphism classes of subgroups of G. We are able to give three alternative descriptions of this ghost ring. One of them is a direct product of endomorphism rings of permutation modules over outer automorphism groups of subgroups of G. These constructions are given in Section 5. Their main properties are stated in Theorem 5.2, Theorem 5.5 and Theorem 5.7, respectively. The last one of these variations is applied later to fusion systems on finite p-groups. The ghost ring $\tilde{B}^\lhd(G,G)$ carries a natural grading whose component in degree 0 is the subring $\tilde{B}^\Delta(G,G)$. Tensoring with $\mathbb{Q}$ leads to a direct sum decomposition $\mathbb{Q}\tilde{B}^\lhd(G,G) = \mathbb{Q}\tilde{B}^\Delta(G,G) \oplus J(\mathbb{Q}\tilde{B}^\lhd(G,G))$, where the second summand denotes the Jacobson radical.

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Via the mark isomorphism, this leads to a decomposition $\mathbb{Q}B^{\lhd}(G,G) = \mathbb{Q}B^{\Delta}(G,G) \oplus J$, where J denotes the Jacobson radical of $\mathbb{Q}B^{\lhd}(G,G)$. Therefore, there is a natural bijection between the simple modules of $\mathbb{Q}B^{\lhd}(G,G)$ and those of $\mathbb{Q}B^{\Delta}(G,G)$.

In [Bc96a] Bouc defines a category whose objects are the finite groups, whose morphism sets are the groups $B(G,H)$ and whose composition law is the bilinear map mentioned above. Representation groups of finite groups can be considered as additive functors (biset functors) on this category. The structure of a biset functor was a key tool in the classification of endo-permutation modules of p-groups, achieved by Bouc, Thévenaz, Carlson, and others. The category of left-free (respectively, bifree) biset functors is equivalent to a module category of the algebra $A^{\lhd} := \bigoplus_{G,H} B^{\lhd}(G,H)$ (respectively $A^{\Delta} := \bigoplus_{G,H} B^{\Delta}(G,H)$), endowed with a natural multiplication. Through the above mark homomorphisms these algebras embed into the ghost algebras $\tilde{A}^{\lhd}$ (respectively $\tilde{A}^{\Delta}$) whose multiplicative structure is much cleaner. Over the rational numbers we even obtain isomorphisms of algebras shedding new light on the biset functors. For instance, known results about the semisimplicity of the bifree-biset-functor category, cf. [W, Corollary 9.2], become very clear from this point of view, cf. Remark 5.10. Turning to left-free biset functors, one can see an obvious structure of graded algebra on $\mathbb{Q}\tilde{A}^{\lhd}$, which is not apparent on $\mathbb{Q}A^{\lhd}$. This leads to a filtration on any left-free biset functor over $\mathbb{Q}$, cf. Remark 6.9.

All the above results we actually state and prove in vastly greater generality. For exposition reasons we have only described them here in the case of the bifree and left-free double Burnside groups. More generally, one can fix a class $\mathcal{D}$ of finite groups and, for every pair (G,H) of groups in $\mathcal{D}$, a set $\mathcal{S}_{G,H}$ of subgroups of $G \times H$ and then consider the Grothendieck group $B^{\mathcal{S}}(G,H)$ of finite bisets whose point stabilizers belong to $\mathcal{S}_{G,H}$. We need to require some axioms on the selection of the sets $\mathcal{S}_{G,H}$, which are stated in Hypothesis 1.13. The reason for formulating results in this generality is to allow, for future applications, more flexible classes of biset functors, and to be able to apply the theory developed so far to fusion systems in the last section of the current paper.

The notion of a fusion system $\mathcal{F}$ on a finite p-group originated from work of Puig about 40 years ago, and has recently got a lot of attention in modular representation theory of finite groups, since every block defines a fusion system on a defect group, in algebraic topology (cf. [BLO03] and [BLO04]), and in abstract group theory (cf. [A] and [AC]) as a potential new avenue to the classification of finite simple groups. The saturated fusion systems play a particularly fundamental role. In this paper we observe (cf. Theorem 7.3) that the set of fusion systems on a finite p-group S is isomorphic (as a partially ordered set) to the set of systems of subgroups of $S \times S$ satisfying the axioms in Hypothesis 1.13. We propose to study the associated subring $B^{\mathcal{F}}(S,S)$ of the bifree double Burnside ring $B^{\Delta}(S,S)$ as an algebraic invariant of a fusion system. In [RS], Ragnarsson and Stancu described a remarkable bijection between the set of saturated fusion systems on S and a certain set of idempotents in $\mathbb{Z}_{(p)}B^{\Delta}(S,S)$. Using the mark homomorphism for $B^{\Delta}(S,S)$, we are able to explicitly determine these idempotents as elements in the ghost algebra $\mathbb{Q}\tilde{B}^{\Delta}(S,S)$, and extend the bijection of Ragnarsson–Stancu to a bijection between the set of all fusion systems on S and a set of idempotents in $\mathbb{Q}\tilde{B}^{\Delta}(S,S)$, cf. Theorem 7.15. In the end, we consider the class of fusion systems whose associated idempotents are mapped, under the mark homomorphism, to idempotents in $\mathbb{Z}_{(p)}\tilde{B}^{\Delta}(S,S)$. By definition, this class contains the class of saturated fusion systems. We show that it is strictly bigger (cf. Example 7.18) and that it shares some properties with the class of saturated fusion systems (cf. Proposition 7.19).

The present paper is arranged as follows: in Section 1 we introduce the necessary notions and recall the necessary results on the double Burnside group and biset functors. In Section 2 we show how one can express the fixed points of the tensor product of two bisets X and Y in terms of fixed points of X and fixed

points of Y. This has an immediate application, expressing the Brauer construction applied to a tensor product of p-permutation bimodules M and N in terms of Brauer constructions of M and N, which is carried out in Section 3. In Section 4 we introduce the construction of the ghost group $\tilde{B}^{\triangleleft}(G, H)$ and the mark homomorphism $\rho_{G,H} \colon B^{\triangleleft}(G, H) \to \tilde{B}^{\triangleleft}(G, H)$, and we prove their main properties. Section 5 continues with a closer study of the group $B^{\Delta}(G, H)$, for which we introduce an alternative ghost group consisting of homomorphism groups and an alternative mark homomorphism $\sigma_{G,H}$. This leads to the semisimplicity results regarding $\mathbb{Q}B^{\Delta}(G, G)$ and related biset functor categories, and to the determination of simple modules of $\mathbb{Q}B^{\Delta}(G, G)$ and of simple biset functors. In Section 6 we show that $\mathbb{Q}B^{\triangleleft}(G, H)$ has a natural grading, turning $\mathbb{Q}B^{\triangleleft}(G, G)$ and the biset algebra $\mathbb{Q}A^{\triangleleft}$ into graded $\mathbb{Q}$-algebras. This enables us to see that the simple modules for $\mathbb{Q}B^{\triangleleft}(G, G)$ (respectively $\mathbb{Q}A^{\triangleleft}$) are the same as for $\mathbb{Q}B^{\Delta}(G, G)$ (respectively $\mathbb{Q}A^{\Delta}$). Finally, in Section 7 we give an application to fusion systems.

1   Bisets and the double Burnside group

Throughout this section, G, H, and K denote finite groups. Moreover, R denotes an associative unitary commutative ring. We recall the definition and basic properties of (G, H)-bisets and their Grothendieck group, the double Burnside group $B(G, H)$. As a convention throughout this paper, G-sets without further specification will always be assumed to be *finite left G-sets*. We refer the reader to [Bc96a], [Bc00], [Bc10], and [CR, §80] for more details concerning the results of this section.

1.1 *Notation.* We will write $U \leqslant G$ to indicate that U is a subgroup of G. For $g \in G$ we denote the inner automorphism $x \mapsto gxg^{-1}$ of G by c_g, and for $U \leqslant G$ we set ${}^{g}U := gUg^{-1}$ and $U^g := g^{-1}Ug$. If U and V are subgroups of G we write $U \leqslant_G V$ if U is G-conjugate to a subgroup of V, and we write $U =_G V$ if U and V are G-conjugate. Moreover, we denote by $\mathrm{Hom}_G(U, V)$ the set of group homomorphisms $\varphi \colon U \to V$ such that there exists some $g \in G$ with $\varphi(u) = c_g(u)$ for all $u \in U$. We also write $\mathrm{Aut}_G(U)$ instead of $\mathrm{Hom}_G(U, U)$. Note that the homomorphism $N_G(U) \to \mathrm{Aut}(U)$, $g \mapsto c_g$, induces an isomorphism $N_G(U)/C_G(U) \to \mathrm{Aut}_G(U)$. Here, $N_G(U)$ and $C_G(U)$ denote the normalizer and centralizer of U in G, respectively. We write Σ_G for the set of subgroups of G. If T is any other group then $\Sigma_G(T)$ denotes the set of subgroups of G that are isomorphic to T.

For any finite subset C of an abelian group or a module we write C^+ for the sum of the elements of C.

The set of positive (respectively, non-negative) integers will be denoted by $\mathbb{N}$ (respectively, $\mathbb{N}_0$).

1.2 *Bisets.* Recall that a (G, H)-*biset* is a *finite* set X, endowed with a left G-action and a right H-action that commute with each other. Together with the (G, H)-*equivariant* maps, the (G, H)-bisets form a category. Every $G \times H$-set X can be regarded as a (G, H)-biset and vice versa, by defining

$$gxh := (g, h^{-1})x \quad \text{and} \quad (g, h)x := gxh^{-1}$$

for $x \in X$, $g \in G$, $h \in H$. We freely use these identifications without further notice. Thus, if $L \leqslant G \times H$ and if X is a (G, H)-biset we can speak of the *L-fixed points* X^L of X and of the $G \times H$-orbits (or just *orbits*) of X.

Note that the group G is an (H, K)-biset for any two subgroups H and K of G, by using left and right multiplication.

A (G, H)-biset is called *left-free* if the left G-action is free (i.e., if every element has trivial G-stabilizer), it is called *right-free* if the right H-action is free, and it is called *bifree* if it is left-free and right-free.

1.3 *Tensor product of bisets.* Let X be a (G, H)-biset and let Y be an (H, K)-biset. The cartesian product $X \times Y$ becomes a (G, K)-biset, by setting $g(x, y)k := (gx, yk)$ for $x \in X$, $y \in Y$, $g \in G$, and $k \in K$. Moreover, $X \times Y$ is a left H-set via $h(x, y) := (xh^{-1}, hy)$. The H-action commutes with the $G \times K$-action, and the set $X \times_H Y$ of H-orbits of $X \times Y$ inherits a (G, K)-biset structure. The H-orbit of the element $(x, y) \in X \times Y$ will be denoted by $x \times_H y \in X \times_H Y$. If L is another group and Z is a (K, L)-biset then $(X \times_H Y) \times_K Z \cong X \times_H (Y \times_K Z)$ as (G, L)-bisets, under $(x \times_H y) \times_K z \mapsto x \times_H (y \times_K z)$. Note also that for a (G, H)-biset X one has isomorphisms

$$ G \times_G X \to X, \ g \times_G x \mapsto gx, \quad \text{and} \quad X \times_H H \to X, \ x \times_H h \mapsto xh \,, $$

of (G, H)-bisets.

1.4 *Opposite biset.* For a (G, H)-biset X we denote by X° its *opposite* biset. Thus, X° is an (H, G)-biset whose underlying set is equal to X and whose biset structure is given by $hx^\circ g := (g^{-1}xh^{-1})^\circ$, for $x \in X$, $g \in G$, and $h \in H$. Here, we write x° if we view the element x of X as an element in X°. Note that X and $(X^\circ)^\circ$ are isomorphic as (G, H)-bisets. If $G = H$ and X is isomorphic to X° as (G, G)-biset then we say that X is *symmetric*.

The *opposite* of a subgroup L of $G \times H$ is defined by

$$ L^\circ := \{(h, g) \in H \times G \mid (g, h) \in G \times H\} \,. $$

Note that L° is a subgroup of $H \times G$ and that

$$ (H \times G)/L^\circ \to (G \times H/L)^\circ \,, \quad (h, g)L^\circ \mapsto ((g, h)L)^\circ \,, $$

is an isomorphism of (H, G)-bisets.

1.5 *Subgroups of $G \times H$.* We denote the canonical projections $G \times H \to G$ and $G \times H \to H$ by p_1 and p_2, respectively. For a subgroup L of $G \times H$ we also set

$$ k_1(L) := \{g \in G \mid (g, 1) \in L\} \quad \text{and} \quad k_2(L) := \{h \in H \mid (1, h) \in L\} \,. $$

Then $k_i(L) \trianglelefteq p_i(L)$ for $i = 1, 2$, and the projection p_i induces an isomorphism $\bar{p}_i \colon L/(k_1(L) \times k_2(L)) \to p_i(L)/k_i(L)$. Thus,

$$ \eta_L = \bar{p}_1 \circ \bar{p}_2^{-1} \colon p_2(L)/k_2(L) \xrightarrow{\sim} p_1(L)/k_1(L) $$

is an isomorphism satisfying $\eta_L(hk_2(L)) = gk_1(L)$ for all $(g, h) \in L$. This construction defines a bijection between the set of subgroups of $G \times H$ and the set of quintuples $(A_1, B_1, \eta, A_2, B_2)$, where $A_1 \trianglelefteq B_1 \leqslant G$, $A_2 \trianglelefteq B_2 \leqslant H$, and $\eta \colon B_2/A_2 \xrightarrow{\sim} B_1/A_1$ is an isomorphism. The inverse of this bijection assigns to a quintuple $(A_1, B_1, \eta, A_2, B_2)$ the subgroup $\{(g, h) \in G \times H \mid \eta(hA_2) = gA_1\}$ of $G \times H$.

In this paper we are mostly interested in the set $\lhd_{G,H}$ of subgroups L of $G \times H$ with $k_1(L) = 1$. These are related to left-free bisets by the next proposition. As a special case of the above (with $A_1 = 1$) these groups can be described as follows. Let $E_{G,H}$ denote the set of triples (U, α, V), where $U \leqslant G$, $V \leqslant H$ and $\alpha \colon V \to U$ is an epimorphism. For $(U, \alpha, V) \in E_{G,H}$ we set

$$\lhd(U, \alpha, V) := \{(\alpha(h), h) \in G \times H \mid h \in V\} \,.$$

Note that if $L = \lhd(U, \alpha, V)$ then $p_1(L) = U$, $p_2(L) = V$, $k_1(L) = 1$ and $k_2(L) = \ker(\alpha)$. This construction defines a bijection

$$E_{G,H} \xrightarrow{\sim} \lhd_{G,H} \,, \quad (U, \alpha, V) \mapsto \lhd(U, \alpha, V) \,. \tag{1}$$

The group $G \times H$ acts on both sets via conjugation. More precisely, on $E_{G,H}$ it acts via

$$^{(g,h)}(U, \alpha, V) := (\, ^gU, c_g \alpha c_h^{-1}, \, ^hV) \,.$$

With this definition, the bijection in (1) is $G \times H$-equivariant.

If $\alpha \colon V \to U$ from above is an isomorphism then we call the subgroup $\lhd(U, \varphi, V)$ a *twisted diagonal* subgroup. In this case we sometimes write $\Delta(U, \alpha, V)$ to indicate that α is an isomorphism. The set of twisted diagonal subgroups of $G \times H$ will be denoted by $\Delta_{G,H}$. Note that a subgroup L of $G \times H$ is twisted diagonal if and only if $k_1(L) = 1$ and $k_2(L) = 1$. The $G \times H$-equivariant bijection in (1) restricts to a $G \times H$-equivariant bijection

$$I_{G,H} \xrightarrow{\sim} \Delta_{G,H} \,, \quad (U, \alpha, V) \mapsto \Delta(U, \alpha, V) \,, \tag{2}$$

where $I_{G,H}$ denotes the set of elements $(U, \alpha, V) \in E_{G,H}$ such that α is an isomorphism.

As a particular case, if $G = H$ and $U \leqslant G$ then we set $\Delta(U) := \Delta(U, \mathrm{id}, U)$.

The following two propositions are easy consequences of the definitions; we leave the proofs to the reader. We denote the stabilizer of an element x of a G-set X by $\mathrm{stab}_G(x)$. If $U \lhd V \leqslant G$ then we write $C_G(U, V)$ for the set of all elements $g \in N_G(U) \cap N_G(V)$ for which the conjugation map c_g induces the identity on V/U.

1.6 Proposition *Let X be a (G, H)-biset. Then X is left-free if and only if $k_1(\mathrm{stab}_{G \times H}(x)) = 1$ for all $x \in X$, and X is right-free if and only if $k_2(\mathrm{stab}_{G \times H}(x)) = 1$ for all $x \in X$. Thus, X is bifree if and only if $\mathrm{stab}_{G \times H}(x)$ is a twisted diagonal subgroup of $G \times H$ for all $x \in X$.*

1.7 Proposition (a) *Let $L \leqslant G \times H$ be arbitrary. Assume that either (i) $\mathcal{A} \subseteq G$ is a transversal for $G/k_1(L)$ and $\mathcal{B} \subseteq H$ is a transversal for $H/p_2(L)$, or that (ii) $\mathcal{A} \subseteq G$ is a transversal for $G/p_1(L)$ and $\mathcal{B} \subseteq H$ is a transversal for $H/k_2(L)$. Then, $\mathcal{A} \times \mathcal{B} \subseteq G \times H$ is a transversal for $G \times H/L$.*

(b) *Let $L \leqslant G \times H$ be arbitrary. Then, for $i = 1, 2$,*

$$k_i(N_{G \times H}(L)) = C_G(k_i(L), p_i(L)) \,.$$

In particular, if $U \leqslant G$, $V \leqslant H$ and if $\alpha \colon V \to U$ is an epimorphism then

$$k_1(N_{G \times H}(\lhd(U, \alpha, V))) = C_G(U) \quad \text{and} \quad k_2(N_{G \times H}(\lhd(U, \alpha, V))) = C_H(\ker(\alpha), V) \,.$$

In particular, one has $C_H(V) \lhd k_2(N_{G \times H}(\lhd(U, \alpha, V)))$.

(c) Let $U \leqslant G$, $V \leqslant H$, and let $\alpha\colon V \to U$ be an isomorphism. Then

$$k_1(N_{G \times H}(\Delta(U, \alpha, V))) = C_G(U), \quad k_2(N_{G \times H}(\Delta(U, \alpha, V))) = C_H(V),$$
$$p_1(N_{G \times H}(\Delta(U, \alpha, V))) = \{g \in N_G(U) \mid \exists h \in N_H(V)\colon c_g = \alpha c_h \alpha^{-1}\} \quad \text{and}$$
$$p_2(N_{G \times H}(\Delta(U, \alpha, V))) = \{h \in N_H(V) \mid \exists g \in N_G(U)\colon c_h = \alpha^{-1} c_g \alpha\}.$$

Usually, the second projection group in Proposition 1.7(c) is denoted by N_α and the first one is denoted by $N_{\alpha^{-1}}$. We will also use this notation in Section 7 in connection with fusion systems.

1.8 The *double Burnside group* $B(G, H)$ of G and H is defined as the Grothendieck group of the category of (G, H)-bisets. Using the identification in 1.2, we may identify $B(G, H)$ with the Burnside group $B(G \times H)$. The group $B(G, H)$ is defined as the factor group F/U, where F is the free abelian group on the set of isomorphism classes $\{X\}$ of (G, H)-bisets X, and U is generated by the elements $\{X \coprod X'\} - \{X\} - \{X'\}$, with arbitrary (G, H)-bisets X and X'. Here, $X \coprod X'$ denotes the coproduct (or disjoint union) of X and X'. The coset of the element $\{X\} \in F$ will be denoted by $[X] \in B(G, H)$. Thus $[X \coprod X'] = [X] + [X']$. If $\mathcal{L}$ is a transversal of the conjugacy classes of subgroups of $G \times H$ then $\{[G \times H/L] \mid L \in \mathcal{L}\}$ is a $\mathbb{Z}$-basis, the *standard basis*, of $B(G, H)$. One has $[X] = [X'] \in B(G, H)$ if and only if X and X' are isomorphic (G, H)-bisets.

The tensor product construction for bisets in 1.3 induces a bilinear map

$$- \cdot_H -\colon B(G, H) \times B(H, K) \to B(G, K), ([X], [Y]) \mapsto [X \times_H Y],$$

where X denotes a (G, H)-biset and Y denotes an (H, K)-biset. In the case that $G = H = K$, this map defines a multiplication on $B(G, G)$ establishing a ring structure with identity element $[G] = [G \times G/\Delta(G)]$. This ring is called the *double Burnside ring* of G.

The construction of the opposite biset induces a group isomorphism

$$-^\circ\colon B(G, H) \to B(H, G), \quad [X] \mapsto [X^\circ],$$

satisfying

$$([X] \cdot_H [Y])^\circ = [Y]^\circ \cdot_H [X]^\circ \in B(K, G) \quad \text{and} \quad ([X]^\circ)^\circ = [X] \in B(G, H),$$

for every (G, H)-biset X and every (H, K)-biset Y. In particular, if $G = H = K$, this implies that $-^\circ\colon B(G, G) \to B(G, G)$ is an anti-involution of $B(G, G)$.

1.9 *The $*$-product of subgroups.* The following proposition, due to Bouc, gives an explicit description of the bilinear map $- \cdot_H -$ on the standard basis elements. It requires the following notation: for subgroups $L \leqslant G \times H$ and $M \leqslant H \times K$ one defines the subgroup

$$L * M \leqslant G \times K$$

as the set of all pairs $(g, k) \in G \times K$ for which there exists some $h \in H$ such that $(g, h) \in L$ and $(h, k) \in M$. Viewing L as a relation between G and H, and M as a relation between H and K, the subgroup $L * M$ is the composition of these two relations. Note that

$$(L * M)^\circ = M^\circ * L^\circ$$

as subgroups of $K \times G$. We emphasize that, in general, the double Burnside ring $B(G, G)$ is not commutative, as can be easily seen from the following proposition.

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1.10 Proposition (cf. [Bc10, 2.3.24]) *For $L \leqslant G \times H$ and $M \leqslant H \times K$ one has*

$$[(G \times H)/L] \cdot_H [(H \times K)/M] = \sum_{h \in [p_2(L) \backslash H / p_1(M)]} [(G \times K)/(L * {}^{(h,1)}M)] \in B(G, K),$$

where $[p_2(L) \backslash H / p_1(M)]$ denotes a set of double coset representatives.

1.11 *Classical ghost group and mark homomorphism.* Recall from [CR, Proposition 80.12] that for every subgroup $L \leqslant G \times H$, one has a group homomorphism $\Phi_L \colon B(G, H) \to \mathbb{Z}$, $[X] \mapsto |X^L|$, with the following properties: if L and L' are conjugate subgroups of $G \times H$ then $\Phi_L = \Phi_{L'}$, and if $\mathcal{L}$ denotes a transversal of the conjugacy classes of subgroups of $G \times H$ then the map

$$\Phi = (\Phi_L)_{L \in \mathcal{L}} \colon B(G, H) \to \prod_{L \in \mathcal{L}} \mathbb{Z}, \quad [X] \mapsto (|X^L|)_{L \in \mathcal{L}},$$

is an injective group homomorphism with finite cokernel. It is called the *classical mark homomorphism*, and its codomain is called the *classical ghost group* of $B(G, H)$. One of the goals of this paper is to construct ghost groups and mark homomorphisms for $B^{\lhd}(G, H)$ that naturally come with bilinear maps that correspond, under the mark homomorphism, to the tensor product of bisets.

For any commutative ring R, the map Φ induces an R-module homomorphism

$$\Phi \colon RB(G, H) \to \prod_{L \in \mathcal{L}} R, \tag{3}$$

where $RB(G, H) := R \otimes_{\mathbb{Z}} B(G, H)$ will often be identified with the free R-module with basis $[G \times H/L]$, $L \in \mathcal{L}$. If $|G \times H|$ is invertible in R then (3) is an R-module isomorphism. Moreover, if R is a field of characteristic 0 then we can view $B(G, H)$ as a subgroup of $RB(G, H)$.

1.12 *The group $B^{\mathcal{S}}(G, H)$.* For a set $\mathcal{S}$ of subgroups of $G \times H$, we define $B^{\mathcal{S}}(G, H)$ as the subgroup of $B(G, H)$ spanned by the standard basis elements $[G \times H/L]$ with $L \in \mathcal{S}$. In the case that $\mathcal{S} = \lhd_{G,H}$ (respectively, $\mathcal{S} = \Delta_{G,H}$) we also use the notation $B^{\lhd}(G, H)$ (respectively, $B^{\Delta}(G, H)$). We call $B^{\lhd}(G, H)$ (respectively, $B^{\Delta}(G, H)$) the *left-free double Burnside group* (respectively, *bifree double Burnside group*) of G and H. Proposition 1.6 justifies the terminology. Clearly one has $B^{\Delta}(G, H) \subseteq B^{\lhd}(G, H)$.

1.13 Hypothesis Assume that we are given a class $\mathcal{D}$ of finite groups and that for any two groups G and H belonging to $\mathcal{D}$ we are given a subset $\mathcal{S}_{G,H}$ of subgroups of $G \times H$. We often write $\mathcal{S}$ for the collection of sets $\mathcal{S}_{G,H}$, $G, H \in \mathcal{D}$. For any groups G and H in $\mathcal{D}$, we define the R-module $RB^{\mathcal{S}_{G,H}}(G, H)$, or for short $RB^{\mathcal{S}}(G, H)$, as in 1.12 above. In subsequent results we will put further restrictions on the sets $\mathcal{S}_{G,H}$. For this, we say that $(\mathcal{D}, \mathcal{S})$ satisfies Condition (I) if the following hold:
    (i) For all $G, H \in \mathcal{D}$, the set $\mathcal{S}_{G,H}$ is closed under $G \times H$-conjugation.
    (ii) For all $G, H \in \mathcal{D}$, the set $\mathcal{S}_{G,H}$ is closed under taking subgroups.
    (iii) For all $G, H, K \in \mathcal{D}$ and all $L \in \mathcal{S}_{G,H}$ and $M \in \mathcal{S}_{H,K}$ one has $L * M \in \mathcal{S}_{G,K}$.
    (iv) For all $G \in \mathcal{D}$ one has $\Delta(G) \in \mathcal{S}_{G,G}$.
Moreover, we say that $(\mathcal{D}, \mathcal{S})$ satisfies Condition (II) if
    (v) for all $G, H \in \mathcal{D}$ one has $(\mathcal{S}_{G,H})^{\circ} = \mathcal{S}_{H,G}$.

1.14 Proposition *Let $\mathcal{D}$ and $\mathcal{S}_{G,H}$ (for $G, H \in \mathcal{D}$) be as in Hypothesis 1.13, and assume further that $(\mathcal{D}, \mathcal{S})$ satisfies Condition (I) in Hypothesis 1.13. Let $G, H, K \in \mathcal{D}$.*

(a) The bilinear map $- \cdot_H - \colon B(G,H) \times B(H,K) \to B(G,K)$ restricts to a bilinear map

$$- \cdot_H - \colon B^{\mathcal{S}}(G,H) \times B^{\mathcal{S}}(H,K) \to B^{\mathcal{S}}(G,K)\,.$$

In particular, $B^{\mathcal{S}}(G,G)$ is a unitary subring of $B(G,G)$.

(b) Assume that $(\mathcal{D}, \mathcal{S})$ additionally satisfies Condition (II) in Hypothesis 1.13. Then the group isomorphism $-^{\circ} \colon B(G,H) \to B(H,G)$ restricts to an isomorphism

$$B^{\mathcal{S}}(G,H) \to B^{\mathcal{S}}(H,G)\,.$$

(c) If $\mathcal{L}$ is a transversal for the conjugacy classes of subgroups of $G \times H$ and if $\mathcal{L}^{\mathcal{S}} := \mathcal{L} \cap \mathcal{S}_{G,H}$ then the classical mark homomorphism $\Phi \colon B(G,H) \to \prod_{L \in \mathcal{L}} \mathbb{Z}$ restricts to a group monomorphism $B^{\mathcal{S}}(G,H) \to \prod_{L \in \mathcal{L}^{\mathcal{S}}} \mathbb{Z}$. Its cokernel is a finite group of order $\prod_{L \in \mathcal{L}^{\mathcal{S}}} [N_{G \times H}(L) : L]$. In particular, if $|G \times H| \in R^{\times}$ then the latter homomorphism induces an R-module isomorphism

$$RB^{\mathcal{S}}(G,H) \to \prod_{L \in \mathcal{L}^{\mathcal{S}}} R\,.$$

Proof Part (a) follows immediately from the explicit formula in Proposition 1.10 and Hypothesis 1.13(i), (iii), and (iv). Part (b) is immediate from Hypothesis 1.13(v). For Part (c), note that the representing matrix (also called the table of marks) of $\Phi \colon B(G,H) \to \prod_{L \in \mathcal{L}} \mathbb{Z}$ with respect to the standard basis elements in appropriate order is an upper triangular square matrix, with rows and columns both indexed by $\mathcal{L}$ and with diagonal entries $[N_{G \times H}(L) : L]$, $L \in \mathcal{L}$. This follows immediately from the fact that if $L \in \mathcal{L}$ is not conjugate to a subgroup of $L' \in \mathcal{L}$ then $(G \times H/L')^L$ is empty. Omitting rows and columns from this matrix indexed by elements $L \in \mathcal{L}$ that do not belong to $\mathcal{S}_{G,H}$ results again in an upper triangular square matrix whose diagonal entries are units in R, provided that $|G \times H|$ is a unit in R. This matrix is the representing matrix of the restricted morphism considered in (c). This completes the proof. $\square$

1.15 Note that (i)–(iv) in Hypothesis 1.13 are satisfied for $\mathcal{S}_{G,H} = \lhd_{G,H}$, and that (i)–(v) are satisfied for $\mathcal{S}_{G,H} = \Delta_{G,H}$. Therefore, $B^{\lhd}(G,G)$ and $B^{\Delta}(G,G)$ are unitary subrings of $B(G,G)$, which we call the *left-free* and the *bifree double Burnside ring* of G, respectively.

1.16 *Biset functors.* Let $\mathcal{D}$ be a set of finite groups, and for each pair (G,H) of groups in $\mathcal{D}$ let $\mathcal{S}_{G,H}$ be a set of subgroups of $G \times H$ satisfying Condition (I) in Hypothesis 1.13. Moreover, let R be a commutative ring.

(a) Following Bouc, cf. [Bc96a], we can define the category $\mathcal{C}^{\mathcal{D},\mathcal{S}}$ whose objects are the groups in $\mathcal{D}$, whose morphisms are given by $\mathrm{Hom}_{\mathcal{C}^{\mathcal{D},s}}(H,G) = B^{\mathcal{S}}(G,H)$ for $G, H \in \mathcal{D}$, and whose composition is induced by the tensor product construction of bisets. This is an additive category. In this setting, a *biset functor* over R is an additive functor from $\mathcal{C}^{\mathcal{D},\mathcal{S}}$ to the category of left R-modules. Together with natural transformations as morphisms, the biset functors form an abelian category $\mathsf{Func}_R^{\mathcal{D},\mathcal{S}}$.

(b) The category $\mathsf{Func}_R^{\mathcal{D},\mathcal{S}}$ is isomorphic to a module category. This construction works for any additive category in place of $\mathcal{C}^{\mathcal{D},\mathcal{S}}$ and goes back to Gabriel in [Ga, Chapter II], see also [W, §2]. Define the R-module

$$A_R^{\mathcal{D},\mathcal{S}} := \bigoplus_{G,H \in \mathcal{D}} RB^{\mathcal{S}}(G,H)$$

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and define a multiplication on $A_R^{\mathcal{D},\mathcal{S}}$ as follows: if $a \in RB^{\mathcal{S}}(G,H)$ and $b \in RB^{\mathcal{S}}(H',K)$ then set $a \cdot b := a \cdot_H b$ if $H = H'$, and $a \cdot b := 0$ if $H \neq H'$. This way, $A_R^{\mathcal{D},\mathcal{S}}$ is an associative ring. If $R = \mathbb{Z}$ then we also use the notation $A^{\mathcal{D},\mathcal{S}}$. For $G \in \mathcal{D}$, we set $e_G := [G \times G/\Delta(G)]$, the identity element in $RB^{\mathcal{S}}(G,G)$. Then the elements e_G, $G \in \mathcal{D}$, form a set of mutually orthogonal idempotents in $A_R^{\mathcal{D},\mathcal{S}}$. The ring $A_R^{\mathcal{D},\mathcal{S}}$ has an identity if and only if $\mathcal{D}$ is finite; in this case the identity is equal to $\sum_{G \in \mathcal{D}} e_G$. Now consider the category $_{A_R^{\mathcal{D},\mathcal{S}}}\mathsf{Mod}^*$ of all left $A_R^{\mathcal{D},\mathcal{S}}$-modules M with the property that $M = \sum_{G \in \mathcal{D}}(e_G M)$. This is an abelian category and the obvious functor, which sends $F \in \mathsf{Func}_R^{\mathcal{D},\mathcal{S}}$ to the module $M := \bigoplus_{G \in \mathcal{D}} F(G)$, defines a category equivalence. An inverse is given by mapping an $A_R^{\mathcal{D},\mathcal{S}}$-module M to the obvious functor F, defined on objects by $F(G) := e_G M$ for $G \in \mathcal{D}$.

2 Fixed points of products of left-free bisets

Throughout this section, G, H, and K denote finite groups. For a (G,H)-biset X, an (H,K)-biset Y, and a subgroup $N \in \vartriangleleft_{G,K}$, we will give an explicit description of the fixed point set $(X \times_H Y)^N$ in terms of the fixed point sets X^L and Y^M with $L \in \vartriangleleft_{G,H}$ and $M \in \vartriangleleft_{H,K}$.

For any two groups U and V we will abbreviate by $E(U,V)$ the set of epimorphisms $\alpha \colon V \to U$, and by $I(U,V)$ the set of isomorphisms $\alpha \colon V \xrightarrow{\sim} U$.

2.1 Let $U \leqslant G$, $V \leqslant H$, $W \leqslant K$ and assume that $\alpha \in E(U,V)$ and $\beta \in E(V,W)$. We will consider the subgroups $\vartriangleleft(U,\alpha,V) \leqslant G \times H$ and $\vartriangleleft(V,\beta,W) \leqslant H \times K$. Note that $E(U,V)$ (respectively, $I(U,V)$) is a left-free (respectively, bifree) $(\mathrm{Aut}_G(U),\mathrm{Aut}_H(V))$-biset and that $E(V,W)$ (respectively, $I(V,W)$) is a left-free (respectively, bifree) $(\mathrm{Aut}_H(V),\mathrm{Aut}_K(W))$-biset by composition. Now let X be a left-free (G,H)-biset and let Y be a left-free (H,K)-biset. Note that $X^{\vartriangleleft(U,\alpha,V)}$ is a left-free $(C_G(U),C_H(V))$-biset and that $Y^{\vartriangleleft(V,\beta,W)}$ is a left-free $(C_H(V),C_K(W))$-biset, by restriction of the structures of X and Y. In the case that X and Y are bifree and α and β are isomorphisms, these bisets are bifree. Also note that

$$x \in X^{\vartriangleleft(U,\alpha,V)} \iff xv = \alpha(v)x \text{ for all } v \in V.$$

If $x \in X^{\vartriangleleft(U,\alpha,V)}$ and $y \in Y^{\vartriangleleft(V,\beta,W)}$ then $x \times_H y \in (X \times_H Y)^{\vartriangleleft(U,\alpha\beta,W)}$. In fact, for all $w \in W$ one has

$$x \times_H yw = x \times_H \beta(w)y = x\beta(w) \times_H y = \alpha(\beta(w))x \times_H y.$$

Since also $xh \times_H y = x \times_H hy$ for all $h \in C_H(V)$, we obtain a well-defined map

$$\overline{\mu} \colon X^{\vartriangleleft(U,\alpha,V)} \times_{C_H(V)} Y^{\vartriangleleft(V,\beta,W)} \to (X \times_H Y)^{\vartriangleleft(U,\alpha\beta,W)}$$
$$x \times_{C_H(V)} y \mapsto x \times_H y$$

between left-free $(C_G(U),C_K(W))$-bisets. Moreover, the map $\overline{\mu}$ is injective. In fact, if $x,x' \in X^{\vartriangleleft(U,\alpha,V)}$ and $y,y' \in Y^{\vartriangleleft(V,\beta,W)}$ satisfy $x \times_H y = x' \times_H y'$ then there exists some $h \in H$ such that $x' = xh^{-1}$ and $y' = hy$. The fixed point properties of y and y' imply

$$h\beta(w)y = hyw = y'w = \beta(w)y' = \beta(w)hy,$$

and since Y is left-free, we obtain $c_h\beta = \beta$ and then $h \in C_H(V)$. Thus, $x' \times_{C_H(V)} y' = x \times_{C_H(V)} y$, and the injectivity of $\overline{\mu}$ is proved.

2.2 (a) For $U \leqslant G$, $W \leqslant K$, and $\gamma \in E(U, W)$ we denote by $\Gamma_H(U, \gamma, W)$ the set of all triples (α, V, β) with $V \leqslant H$, $\alpha \in E(U, V)$ and $\beta \in E(V, W)$ such that $\alpha\beta = \gamma$. In other words, $\Gamma_H(U, \gamma, W)$ consists of all factorizations of γ as two epimorphisms via subgroups of H.

(b) Note that H acts on $\Gamma_H(U, \gamma, W)$ by

$$h(\alpha, V, \beta) := (\alpha c_h^{-1}, \, {}^h V, c_h \beta)$$

and that the stabilizer of (α, V, β) is equal to $C_H(V)$.

Note also that if (α, V, β) and $(\tilde{\alpha}, \tilde{V}, \tilde{\beta})$ lie in the same H-orbit of $\Gamma_H(U, \gamma, W)$ and if $h \in H$ satisfies $(\tilde{\alpha}, \tilde{V}, \tilde{\beta}) = h(\alpha, V, \beta) = (\alpha c_h^{-1}, \, {}^h V, c_h \beta)$ then one has an isomorphism of $(C_G(U), C_K(W))$-bisets,

$$\varphi_h \colon X^{\lhd(U,\alpha,V)} \times Y^{\lhd(V,\beta,W)} \xrightarrow{\sim} X^{\lhd(U,\tilde{\alpha},\tilde{V})} \times Y^{\lhd(\tilde{V},\tilde{\beta},W)}, \quad (x,y) \mapsto (xh^{-1}, hy),$$

which induces an isomorphism

$$\overline{\varphi}_h \colon X^{\lhd(U,\alpha,V)} \times_{C_H(V)} Y^{\lhd(V,\beta,W)} \xrightarrow{\sim} X^{\lhd(U,\tilde{\alpha},\tilde{V})} \times_{C_H(\tilde{V})} Y^{\lhd(\tilde{V},\tilde{\beta},W)},$$

$$x \times_{C_H(V)} y \mapsto xh^{-1} \times_{C_H(\tilde{V})} hy,$$

of $(C_G(U), C_K(W))$-bisets such that $\overline{\mu} \circ \overline{\varphi}_h = \overline{\mu}$. The isomorphism $\overline{\varphi}_h$ does not depend on the choice of the element $h \in H$. In fact, if also $h'(\alpha, V, \beta) = (\tilde{\alpha}, \tilde{V}, \tilde{\beta})$ then $h' = ch$ for some $c \in C_H(\tilde{V})$.

The following result will be crucial for studying the ring structures of $B^{\lhd}(G, G)$ and $B^{\Delta}(G, G)$.

2.3 Theorem *Let $U \leqslant G$ and $W \leqslant K$, and let $\gamma \in E(U, W)$. Let $\Gamma_H(U, \gamma, W)$ be defined as in 2.2 and let $\tilde{\Gamma}_H(U, \gamma, W) \subseteq \Gamma_H(U, \gamma, W)$ be a set of representatives of the H-orbits of $\Gamma_H(U, \gamma, W)$. Then the maps $\overline{\mu}$ from 2.1 induce an isomorphism of $(C_G(U), C_K(W))$-bisets*

$$\coprod_{(\alpha,V,\beta) \in \tilde{\Gamma}_H(U,\gamma,W)} X^{\lhd(U,\alpha,V)} \times_{C_H(V)} Y^{\lhd(V,\beta,W)} \xrightarrow{\sim} (X \times_H Y)^{\lhd(U,\gamma,W)}.$$

Moreover, if $\tilde{\Sigma}_H \subseteq \Sigma_H$ is a transversal of the H-conjugacy classes of subgroups of H then

$$|(X \times_H Y)^{\lhd(U,\gamma,W)}| = \sum_{V \leqslant H} |H|^{-1} \sum_{\substack{(\alpha,\beta) \in E(U,V) \times E(V,W) \\ \alpha\beta = \gamma}} |X^{\lhd(U,\alpha,V)}| \cdot |Y^{\lhd(V,\beta,W)}|$$

$$= \sum_{V \in \tilde{\Sigma}_H} |N_H(V)|^{-1} \sum_{\substack{(\alpha,\beta) \in E(U,V) \times E(V,W) \\ \alpha\beta = \gamma}} |X^{\lhd(U,\alpha,V)}| \cdot |Y^{\lhd(V,\beta,W)}|.$$

Proof We first show that the map in the theorem is surjective. So let $x \in X$, $y \in Y$ be such that $x \times_H y \in (X \times_H Y)^{\lhd(U,\gamma,W)}$. Then $x \times_H yw = \gamma(w)x \times_H y$ for all $w \in W$. This implies that, for every $w \in W$, there exists an element $h_w \in H$ such that

$$xh_w = \gamma(w)x \quad \text{and} \quad h_w y = yw.$$

Since Y is left-free, h_w is uniquely determined by w. Thus, we obtain a function $\tilde{\beta} \colon W \to H$ such that

$$x\tilde{\beta}(w) = \gamma(w)x \quad \text{and} \quad \tilde{\beta}(w)y = yw \tag{4}$$

for all $w \in W$. Moreover, for $w, w' \in W$ we have

$$\tilde{\beta}(ww')y = yww' = \tilde{\beta}(w)yw' = \tilde{\beta}(w)\tilde{\beta}(w')y \,,$$

and since Y is left-free, we see that $\tilde{\beta}$ is a group homomorphism. We set $\tilde{V} := \tilde{\beta}(W)$ and have $\tilde{\beta} \in E(\tilde{V}, W)$. By the second equation in (4), we have $y \in Y^{\lhd(\tilde{V}, \tilde{\beta}, W)}$. Next we define a function $\tilde{\alpha} \colon \tilde{V} \to U$ as follows. For $\tilde{v} \in \tilde{V}$, choose $w \in W$ with $\tilde{\beta}(w) = \tilde{v}$, and set $\tilde{\alpha}(\tilde{v}) := \gamma(w)$. This is well defined. For if also $w' \in W$ satisfies $\tilde{\beta}(w') = \tilde{v}$ then, by the first equation in (4),

$$\gamma(w)x = x\tilde{\beta}(w) = x\tilde{\beta}(w') = \gamma(w')x \,.$$

Since X is left-free, we obtain $\gamma(w) = \gamma(w')$. By construction, $\tilde{\alpha}(\tilde{V}) = \gamma(W) = U$ and $\gamma = \tilde{\alpha}\tilde{\beta}$. Since γ and $\tilde{\beta}$ are surjective homomorphisms, also $\tilde{\alpha}$ is a surjective homomorphism. Now, the first equation in (4) implies that $x \in X^{\lhd(U, \tilde{\alpha}, \tilde{V})}$. Since $(\tilde{\alpha}, \tilde{V}, \tilde{\beta})$ is an element in $\Gamma_H(U, \gamma, W)$, there exist $h \in H$ and $(\alpha, V, \beta) \in \tilde{\Gamma}_H(U, \gamma, W)$ such that $(\alpha, V, \beta) = h(\tilde{\alpha}, \tilde{V}, \tilde{\beta})$. This implies that $xh^{-1} \in X^{\lhd(U, \alpha, V)}$ and $hy \in Y^{\lhd(V, \beta, W)}$. Thus, $x \times_H y = xh^{-1} \times_H hy$ lies in the image of the map in the theorem.

Next we show that the map in the theorem is injective. Let $(\alpha, V, \beta), (\tilde{\alpha}, \tilde{V}, \tilde{\beta}) \in \tilde{\Gamma}_H(U, \gamma, W)$ and let $x \in X^{\lhd(U, \alpha, V)}$, $y \in Y^{\lhd(V, \beta, W)}$, $\tilde{x} \in X^{\lhd(U, \tilde{\alpha}, \tilde{V})}$, and $\tilde{y} \in Y^{\lhd(\tilde{V}, \tilde{\beta}, W)}$ be such that $x \times_H y = \tilde{x} \times_H \tilde{y}$. By the injectivity of the map $\overline{\mu}$ in 2.1, it suffices to show that $(\alpha, V, \beta) = (\tilde{\alpha}, \tilde{V}, \tilde{\beta})$, or, equivalently, that $h(\alpha, V, \beta) = (\tilde{\alpha}, \tilde{V}, \tilde{\beta})$ for some $h \in H$. Now, since $x \times_H y = \tilde{x} \times_H \tilde{y}$, there exists $h \in H$ such that $\tilde{x} = xh^{-1}$ and $\tilde{y} = hy$. Moreover, for all $w \in W$, we have

$$\beta(w)y = yw \quad \text{and} \quad \tilde{\beta}(w)hy = \tilde{\beta}(w)\tilde{y} = \tilde{y}w = hyw \,.$$

These two equations imply $h^{-1}\tilde{\beta}(w)hy = yw = \beta(w)y$. Since Y is left-free, we obtain $\tilde{\beta} = c_h\beta$ and $\tilde{V} = \tilde{\beta}(W) = (c_h\beta)(W) = {}^h V$. In order to see that $\tilde{\alpha} = \alpha c_h^{-1}$, let $\tilde{v} \in \tilde{V}$ and choose $w \in W$ such that $\tilde{\beta}(w) = \tilde{v}$. Then

$$\tilde{\alpha}(\tilde{v}) = (\tilde{\alpha}\tilde{\beta})(w) = \gamma(w) = (\alpha\beta)(w) = (\alpha c_h^{-1}\tilde{\beta})(w) = (\alpha c_h^{-1})(\tilde{v}) \,.$$

This implies that $(\tilde{\alpha}, \tilde{V}, \tilde{\beta}) = h(\alpha, V, \beta)$ and completes the proof of the injectivity of the map in the theorem.

Finally, we will show the equations in the theorem. First note that

$$\left|X^{\lhd(U, \alpha, V)} \times_{C_H(V)} Y^{\lhd(V, \beta, W)}\right| = |C_H(V)|^{-1} \cdot \left|X^{\lhd(U, \alpha, V)} \times Y^{\lhd(V, \beta, W)}\right| \,,$$

since Y is left-free. Next, recall from 2.2(b), that if (α, V, β) and $(\tilde{\alpha}, \tilde{V}, \tilde{\beta})$ lie in the same H-orbit of $\Gamma_H(U, \gamma, W)$ then $X^{\lhd(U, \alpha, V)} \times Y^{\lhd(V, \beta, W)}$ and $X^{\lhd(U, \tilde{\alpha}, \tilde{V})} \times Y^{\lhd(\tilde{V}, \tilde{\beta}, W)}$ are in bijective correspondence. Since the H-orbit of (α, V, β) has size $[H : C_H(V)]$, these two facts and the isomorphism in the theorem imply the first equation. The second equation is immediate. $\qquad\square$

2.4 Remark (a) Note that the left-hand side of the isomorphism in Theorem 2.3 does not depend on the choice of the set $\tilde{\Gamma}_H(U, \gamma, W)$, in the sense that any other choice would lead to *canonically* isomorphic components, the isomorphism being given by the maps $\overline{\varphi}_h$, cf. 2.2(b).

(b) If the epimorphism γ in Theorem 2.3 is an isomorphism and if (α, V, β) is an element in $\Gamma_H(U, \gamma, W)$, then also α and β are isomorphisms, since $\gamma = \alpha\beta$. In particular, V is isomorphic to U and W. Therefore, Theorem 2.3 implies immediately the following result.

2.5 Theorem *Let $U \leqslant G$ and $W \leqslant K$ be isomorphic subgroups of G and K, respectively, and let $\gamma \in I(U,W)$ be an isomorphism between them. Let $\Gamma_H(U,\gamma,W)$ be defined as in 2.2 and let $\tilde{\Gamma}_H(U,\gamma,W) \subseteq \Gamma_H(U,\gamma,W)$ be a set of representatives of the H-orbits of $\Gamma_H(U,\gamma,W)$. Then the maps $\overline{\mu}$ from 2.1 induce an isomorphism of $(C_G(U), C_K(W))$-bisets*

$$\coprod_{(\alpha,V,\beta)\in\tilde{\Gamma}_H(U,\gamma,W)} X^{\Delta(U,\alpha,V)} \times_{C_H(V)} Y^{\Delta(V,\beta,W)} \xrightarrow{\sim} (X \times_H Y)^{\Delta(U,\gamma,W)} \,.$$

Moreover, if $\tilde{\Sigma}_H(U) \subseteq \Sigma_H(U)$ is a transversal of the H-conjugacy classes of $\Sigma_H(U)$ then

$$|(X \times_H Y)^{\Delta(U,\gamma,W)}| = \sum_{V \leqslant H} |H|^{-1} \sum_{\substack{(\alpha,\beta)\in I(U,V)\times I(V,W) \\ \alpha\beta=\gamma}} |X^{\Delta(U,\alpha,V)}| \cdot |Y^{\Delta(V,\beta,W)}|$$

$$= \sum_{V \in \tilde{\Sigma}_H(U)} |N_H(V)|^{-1} \sum_{\substack{(\alpha,\beta)\in I(U,V)\times I(V,W) \\ \alpha\beta=\gamma}} |X^{\Delta(U,\alpha,V)}| \cdot |Y^{\Delta(V,\beta,W)}| \,.$$

3 An application to the Brauer constructions of tensor products of p-permutation bimodules

Throughout this section let F be a field of prime characteristic p. Also, G, H, and K will denote finite groups. We will denote by FG the group algebra of G over F, and for any G-set X we will denote by FX the F-vector space with basis X. The left G-action on X induces a left FG-module structure on FX. Similarly, if X is a (G,H)-biset then we obtain an (FG, FH)-bimodule FX. All modules over group algebras will be assumed to have finite F-dimension.

3.1 In this subsection we recall some concepts and results from modular representation theory. We refer the reader to [Br] for the statements concerning p-permutation modules and the Brauer construction, and to [N-T] for the theory of vertices of indecomposable modules.

(a) Similarly as for bisets, we will identify left $F[G \times H]$-modules M with (FG, FH)-bimodules by defining $gmh := (g, h^{-1})m$ and $(g,h)m := gmh^{-1}$, for $m \in M$, $g \in G$ and $h \in H$.

(b) An FG-module that is isomorphic to a direct summand of a module of the form FX, for some G-set X, is called a *p-permutation module*. Thus, we call an (FG, FH)-bimodule M a *p-permutation bimodule* if it is isomorphic to a direct summand of a bimodule of the form FX for a (G,H)-biset X.

(c) It is well known and easy to check that if X is a (G,H)-biset and Y is an (H,K)-biset then the map

$$F[X \times_H Y] \to FX \otimes_{FH} FY \,, \quad x \times_H y \mapsto x \otimes_{FH} y \,,$$

is an isomorphism of (FG, FK)-bimodules.

(d) A *vertex* of an indecomposable FG-module M is a subgroup P of G that is minimal with respect to inclusion and the property that M is isomorphic to a direct summand of the FG-module $FG \otimes_{FP} M$. The set of vertices of M is a single conjugacy class of p-subgroups of G. The vertices of an indecomposable (FG, FH)-bimodule M are considered as subgroups of $G \times H$, by viewing M as an $F[G \times H]$-module.

(e) If M is an indecomposable p-permutation FG-module and P is a vertex of M then M is isomorphic to a direct summand of $F[G/P]$, where G/P is viewed as a G-set.

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(f) For an FG-module M and a p-subgroup P of G, the *Brauer construction* of M with respect to P is defined as

$$M(P) := M^P / \sum_{Q < P} \text{tr}_Q^P(M^Q)$$

where M^P denotes the set of P-fixed points of M and $\text{tr}_Q^P \colon M^Q \to M^P$ denotes the relative trace map, which maps a Q-fixed point m of M to $\sum_{xQ \in P/Q} xm$. The vector space $M(P)$ inherits an $FN_G(P)$-module structure from the FG-module structure of M. Moreover P acts trivially on $M(P)$. It is a well-known fact that if $M(P) \neq 0$ then P is contained in a vertex of some indecomposable direct summand of M.

(g) It is well known that if $M = FX$, for a G-set X, then the composition

$$F(X^P) \to (FX)^P \to M(P)$$

of the canonical maps defines an isomorphism of $FN_G(P)$-modules.

(h) Let M be an (FG, FH)-bimodule and let N be an (FH, FK)-bimodule. Moreover let $U \leqslant G$ and $W \leqslant K$ be p-subgroups and let $\gamma \in E(U, W)$. If $V \leqslant H$ and if $\alpha \in E(U, V)$ and $\beta \in E(V, W)$ satisfy $\alpha\beta = \gamma$ then one has a well-defined bilinear map

$$M^{\lhd(U,\alpha,V)} \times N^{\lhd(V,\beta,W)} \to (M \otimes_{FH} N)^{\lhd(U,\gamma,W)}, \quad (m,n) \mapsto m \otimes n.$$

Moreover, assume that $m = \text{tr}_{\lhd(U',\alpha',V')}^{\lhd(U\alpha,V)}(m')$, with $\lhd(U',\alpha',V') < \lhd(U,\alpha,V)$ and $m' \in M^{\lhd(U',\alpha',V')}$. Then $V' < V$, $U' \leqslant U$ and $\alpha' = \alpha|_{V'}$. We set $W' := \beta^{-1}(V')$ and obtain a bijection $W/W' \to V/V'$, induced by β. We also set $\gamma' := \gamma|_{W'}$ and obtain $(U', \gamma', W') \in E(U', W')$ with $\gamma' = \alpha'\beta|_{W'} = \gamma|_{W'}$, and $\lhd(U',\gamma',W') < \lhd(U,\gamma,W)$. For $n \in N^{\lhd(V,\beta,W)}$ we have

$$m \otimes n = \text{tr}_{\lhd(U',\alpha',V')}^{\lhd(U,\alpha,V)}(m') \otimes n = \sum_{v \in V/V'} \alpha(v)m'v^{-1} \otimes n = \sum_{v \in V/V'} \alpha(v)m' \otimes v^{-1}n$$

$$= \sum_{w \in W/W'} \gamma(w)m' \otimes \beta(w)^{-1}n = \sum_{w \in W/W'} \gamma(w)m' \otimes nw^{-1} = \text{tr}_{\lhd(U',\gamma',W')}^{\lhd(U,\gamma,W)}(m' \otimes n).$$

Similarly, if n is a trace from a proper subgroup of $\lhd(V, \beta, W)$ and m is arbitrary then $m \otimes n$ is again a trace from a proper subgroup of $\lhd(U, \gamma, W)$. Thus, the above map induces a bilinear map

$$M(\lhd(U,\alpha,V)) \times N(\lhd(V,\beta,W)) \to (M \otimes_{FH} N)(\lhd(U,\gamma,W)), \quad (\overline{m}, \overline{n}) \mapsto \overline{m \otimes n}.$$

Since $M(\lhd(U,\alpha,V))$ is an $FN_{G \times H}(\lhd(U,\alpha,V))$-module and $C_G(U) \times C_H(V) \leqslant N_{G \times H}(\lhd(U,\alpha,V))$, we can regard $M(\lhd(U,\alpha,V))$ as $(FC_G(U), FC_H(V))$-bimodule. Similarly, we can view $M(\lhd(V,\beta,W))$ as $(FC_H(V), FC_K(W))$-bimodule and $M(\lhd(U,\gamma,W))$ as $(FC_G(U), FC_K(W))$-bimodule. The above map now induces a homomorphism

$$M(\lhd(U,\alpha,V)) \otimes_{FC_H(V)} N(\lhd(V,\beta,W)) \to (M \otimes_{FH} N)(\lhd(U,\gamma,W)), \quad \overline{m} \otimes \overline{n} \mapsto \overline{m \otimes n},$$

of $(FC_G(U), FC_K(W))$-bimodules. As a special case, if α and β are isomorphisms, we obtain a map

$$M(\Delta(U,\alpha,V)) \otimes_{FC_H(V)} N(\Delta(V,\beta,W)) \to (M \otimes_{FH} N)(\Delta(U,\gamma,W)), \quad \overline{m} \otimes \overline{n} \mapsto \overline{m \otimes n},$$

of $(FC_G(U), FC_K(W))$-bimodules.

3.2 Theorem *Let M be a p-permutation (FG, FH)-bimodule and let N be a p-permutation (FH, FK)-bimodule. Assume that the vertices of every indecomposable direct summand of M and N lie in $\lhd_{G,H}$ and $\lhd_{H,K}$, respectively. Let $U \leqslant G$ and $W \leqslant K$ be p-subgroups, and let $\gamma \in E(U, W)$. Furthermore, let $\tilde{\Gamma}_H(U, \gamma, W)$ be a set of representatives of the H-orbits of $\Gamma_H(U, \gamma, W)$. Then the canonical maps in 3.1(h) induce an isomorphism*

$$\bigoplus_{(\alpha, V, \beta) \in \tilde{\Gamma}_H(U, \gamma, W)} M(\lhd(U, \alpha, V)) \otimes_{FC_H(V)} N(\lhd(V, \beta, W)) \xrightarrow{\sim} (M \otimes_{FH} N)(\lhd(U, \gamma, W)),$$

$$\overline{m} \otimes \overline{n} \mapsto \overline{m \otimes n},$$

of $(FC_G(U), FC_K(W))$-bimodules.

Proof We first note that the left-hand side and the right-hand side can be considered as the evaluation of a functor from the category product of the category of (FG, FH)-bimodules and the category of (FH, FK)-bimodules to the category of $(FC_G(U), FC_K(W))$-bimodules. Moreover, it is easy to see that the map in the theorem gives a natural transformation between these two functors. Both functors respect direct sums in both arguments in a bilinear way. It follows immediately that the map in the theorem is an isomorphism for M and N if and only if it is an isomorphism for every pair of indecomposable direct summands of M and N, respectively. Therefore and by 3.1(e), it suffices to show that the map is an isomorphism in the case where M and N are of the form FX and FY, respectively, for a left-free (G, H)-biset X and a left-free (H, K)-biset Y. But in this case, the map in the theorem is induced by the map in Theorem 2.3, using the canonical identifications from 3.1(c) and (g). $\quad\square$

Remark 2.4(b) and the last sentence in 3.1 imply immediately the following theorem.

3.3 Theorem *Let M be a p-permutation (FG, FH)-bimodule and let N be a p-permutation (FH, FK)-bimodule. Assume that the vertices of every indecomposable direct summand of M and N lie in $\Delta_{G,H}$ and $\Delta_{H,K}$, respectively. Let $U \leqslant G$ and $W \leqslant K$ be isomorphic p-subgroups of G and K, respectively, and let $\gamma \in I(U, W)$. Furthermore, let $\tilde{\Gamma}_H(U, \gamma, W)$ be a set of representatives of the H-orbits of $\Gamma_H(U, \gamma, W)$. Then the canonical maps in 3.1(h) induce an isomorphism*

$$\bigoplus_{(\alpha, V, \beta) \in \tilde{\Gamma}_H(U, \gamma, W)} M(\Delta(U, \alpha, V)) \otimes_{FC_H(V)} N(\Delta(V, \beta, W)) \xrightarrow{\sim} (M \otimes_{FH} N)(\Delta(U, \gamma, W)),$$

$$\overline{m} \otimes \overline{n} \mapsto \overline{m \otimes n},$$

of $(FC_G(U), FC_K(W))$-bimodules.

4 A ghost group $\tilde{B}^{\lhd}(G, H)$ for $B^{\lhd}(G, H)$ and a mark homomorphism

In this section we will construct, for any two finite groups G and H, a ghost group $\tilde{B}^{\lhd}(G, H)$ of the left-free double Burnside group $B^{\lhd}(G, H)$, together with a mark homomorphism $\rho_{G,H}^{\lhd} \colon B^{\lhd}(G, H) \to \tilde{B}^{\lhd}(G, H)$ that is injective and has finite cokernel. For any finite groups G, H, and K, we define a bilinear map $- \cdot_H - \colon \tilde{B}^{\lhd}(G, H) \times \tilde{B}^{\lhd}(H, K) \to \tilde{B}^{\lhd}(G, K)$ that corresponds under the mark homomorphism to the tensor product on the double Burnside groups.

We will adopt a more functorial and general approach by assuming throughout this section that $\mathcal{D}$ is a class of finite groups and that, for any $G, H \in \mathcal{D}$, we are given a set $\mathcal{S}_{G,H} \subseteq \lhd_{G,H}$ of subgroups such that $\mathcal{D}$ together with the system $\mathcal{S}$ of all these sets $\mathcal{S}_{G,H}$ satisfies Condition (I) in Hypothesis 1.13. Note that $\mathcal{D}$ could be an arbitrary set of finite groups, ranging from just one group (as considered in Section 7) to the case of all finite groups. Using a system $\mathcal{S}$ will enable us to carry out the construction of the ghost groups and mark homomorphism at the same time for left-free and bifree double Burnside groups, as well as for the situation of a fusion system, which will be considered in Section 7. The constructions are functorial in $\mathcal{S}$ in the sense that if $\mathcal{S}'$ is a subsystem of $\mathcal{S}$ then $\tilde{B}^{\mathcal{S}'}(G, H)$ is a subset of $\tilde{B}^{\mathcal{S}}(G, H)$ and the mark homomorphism $\rho_{G,H}^{\mathcal{S}'}$ is the restriction of $\rho_{G,H}^{\mathcal{S}}$.

As before, R denotes an associative unitary commutative ring.

4.1 *The group $A^{\mathcal{S}}(G, H)$.* For $G, H \in \mathcal{D}$, we set

$$E_{G,H}^{\mathcal{S}} := \left\{ (U, \alpha, V) \in E_{G,H} \mid \lhd(U, \alpha, V) \in \mathcal{S}_{G,H} \right\}.$$

The $G \times H$-equivariant bijection in (1) restricts, by construction, to a $G \times H$-equivariant bijection

$$E_{G,H}^{\mathcal{S}} \to \mathcal{S}_{G,H}, \quad (U, \alpha, V) \mapsto \lhd(U, \alpha, V). \tag{5}$$

If $\mathcal{S}_{G,H} \subseteq \Delta_{G,H}$ then we also write $I_{G,H}^{\mathcal{S}}$ for $E_{G,H}^{\mathcal{S}}$.

We define $A^{\mathcal{S}}(G, H)$ as the free $\mathbb{Z}$-module with basis $E_{G,H}^{\mathcal{S}}$. In the case that $\mathcal{S}_{G,H} = \lhd_{G,H}$ we write $A^{\lhd}(G, H)$ for $A^{\mathcal{S}}(G, H)$. If $\mathcal{S}_{G,H} = \Delta_{G,H}$ then we also write $A^{\Delta}(G, H)$ for $A^{\mathcal{S}}(G, H)$. We identify the R-module $R \otimes_{\mathbb{Z}} A^{\mathcal{S}}(G, H) = RA^{\mathcal{S}}(G, H)$ with the free R-module with R-basis $E_{G,H}^{\mathcal{S}}$. The $G \times H$-conjugation action on $E_{G,H}^{\mathcal{S}}$ induces a left $R[G \times H]$-module structure on $RA^{\mathcal{S}}(G, H)$.

4.2 Definition For $G, H \in \mathcal{D}$ we define the *ghost group* $\tilde{B}^{\mathcal{S}}(G, H)$ of $B^{\mathcal{S}}(G, H)$ by

$$\tilde{B}^{\mathcal{S}}(G, H) := A^{\mathcal{S}}(G, H)^{G \times H},$$

the set of $G \times H$-fixed points of the $\mathbb{Z}[G \times H]$-module $A^{\mathcal{S}}(G, H)$. If $[U, \alpha, V]_{G \times H}$ denotes the $G \times H$-orbit of the element $(U, \alpha, V) \in E_{G,H}^{\mathcal{S}}$ then the orbit sums $[U, \alpha, V]_{G \times H}^{+}$ form a $\mathbb{Z}$-basis of $\tilde{B}^{\mathcal{S}}(G, H)$. We set $R\tilde{B}^{\mathcal{S}}(G, H) := R \otimes_{\mathbb{Z}} \tilde{B}^{\mathcal{S}}(G, H)$, and identify this R-module in the canonical way with the free R-module with basis $[U, \alpha, V]_{G \times H}^{+}$. We will call this basis the *standard basis* of $R\tilde{B}^{\mathcal{S}}(G, H)$.

We define the *mark homomorphism* $\rho_{G,H}^{\mathcal{S}}$ as the $\mathbb{Z}$-linear map, given by

$$\rho_{G,H}^{\mathcal{S}} \colon B^{\mathcal{S}}(G, H) \to \tilde{B}^{\mathcal{S}}(G, H)$$

$$[X] \mapsto \sum_{(U, \alpha, V) \in E_{G,H}^{\mathcal{S}}} \frac{|X^{\lhd(U, \alpha, V)}|}{|C_G(U)|} (U, \alpha, V),$$

where X is any (G, H)-biset with point stabilizers in $\mathcal{S}_{G,H}$. Note that $|C_G(U)|$ divides $|X^{\lhd(U, \alpha, V)}|$, since $X^{\lhd(U, \alpha, V)}$ is a left-free $C_G(U)$-set, cf. 2.1. Tensoring with R over $\mathbb{Z}$ induces an R-module homomorphism

$$\rho_{G,H}^{\mathcal{S}} \colon RB^{\mathcal{S}}(G, H) \to R\tilde{B}^{\mathcal{S}}(G, H).$$

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4.3 Remark If $\mathcal{S}'$ is another collection of systems of subgroups of $\lhd_{G,H}$, for $G, H \in \mathcal{D}$, and if $\mathcal{S}' \subseteq \mathcal{S}$ (i.e., if $\mathcal{S}'_{G,H} \subseteq \mathcal{S}_{G,H}$ for all $G, H \in \mathcal{D}$) then, for every $G, H \in \mathcal{D}$, the R-module $R\tilde{B}^{\mathcal{S}'}(G,H)$ is a submodule of $R\tilde{B}^{\mathcal{S}}(G,H)$ and the map $\rho_{G,H}^{\mathcal{S}'} \colon RB^{\mathcal{S}'}(G,H) \to R\tilde{B}^{\mathcal{S}'}(G,H)$ is the restriction of the map $\rho_{G,H}^{\mathcal{S}} \colon RB^{\mathcal{S}}(G,H) \to R\tilde{B}^{\mathcal{S}}(G,H)$. In fact, since $\mathcal{S}'_{G,H}$ is closed under taking subgroups, one has $\Phi_L(a) = 0$ for every $a \in RB^{\mathcal{S}'}(G,H)$ and $L \in \mathcal{S}_{G,H} \smallsetminus \mathcal{S}'_{G,H}$.

4.4 *Tensor product on $\tilde{B}^{\mathcal{S}}(G,H)$ and $\tilde{B}^{\mathcal{S}}(H,K)$.* We first define a $\mathbb{Q}$-bilinear map

$$- \cdot_H - \colon \mathbb{Q}A^{\mathcal{S}}(G,H) \times \mathbb{Q}A^{\mathcal{S}}(H,K) \to \mathbb{Q}A^{\mathcal{S}}(G,K) , \tag{6}$$

by setting

$$(U, \alpha, V) \cdot_H (V', \beta, W) := \begin{cases} 0, & \text{if } V \neq V', \\ \frac{|C_H(V)|}{|H|}(U, \alpha\beta, W) & \text{if } V = V'. \end{cases} \tag{7}$$

It will turn out that the unmotivated factor $[H : C_H(V)]^{-1}$ needs to be there in order for the mark homomorphism to translate the tensor product of bisets into this product. For $a \in \mathbb{Q}A^{\mathcal{S}}(G,H)$, $b \in \mathbb{Q}A^{\mathcal{S}}(H,K)$, and $c \in \mathbb{Q}A^{\mathcal{S}}(K,L)$ one has

$$(a \cdot_H b) \cdot_K c = a \cdot_H (b \cdot_K c) . \tag{8}$$

In particular, the vector space $\mathbb{Q}A^{\mathcal{S}}(G,G)$, together with the multiplication $- \cdot_G -$, is a $\mathbb{Q}$-algebra with identity element $\sum_{U \leqslant G}[G : C_G(U)](U, \mathrm{id}_U, U)$. Moreover, for a and b as above, $g \in G$, $h \in H$, and $k \in K$, one has

$$g(a \cdot_H b)k = (ga) \cdot_H (bk) \quad \text{and} \quad (ah) \cdot_H b = a \cdot_H (hb) . \tag{9}$$

The first of these two equations implies that the above bilinear map restricts to a $\mathbb{Q}$-bilinear map

$$- \cdot_H - \colon \mathbb{Q}A^{\mathcal{S}}(G,H)^{G \times H} \times \mathbb{Q}A^{\mathcal{S}}(H,K)^{H \times K} \to \mathbb{Q}A^{\mathcal{S}}(G,K)^{G \times K} . \tag{10}$$

Recall that $\tilde{B}^{\mathcal{S}}(G,H) = A^{\mathcal{S}}(G,H)^{G \times H}$. By the next lemma, this bilinear map restricts to a bilinear map

$$- \cdot_H - \colon \tilde{B}^{\mathcal{S}}(G,H) \times \tilde{B}^{\mathcal{S}}(H,K) \to \tilde{B}^{\mathcal{S}}(G,K) ,$$

which we call the tensor product. Again, by tensoring with R over $\mathbb{Z}$ we obtain an R-bilinear tensor product $- \cdot_H - \colon R\tilde{B}^{\mathcal{S}}(G,H) \times R\tilde{B}^{\mathcal{S}}(H,K) \to R\tilde{B}^{\mathcal{S}}(G,K)$. If $\mathcal{S}' \subseteq \mathcal{S}$, then the tensor product map on $R\tilde{B}^{\mathcal{S}}(G,H) \times R\tilde{B}^{\mathcal{S}}(H,K)$ restricts to the one on $R\tilde{B}^{\mathcal{S}'}(G,H) \times R\tilde{B}^{\mathcal{S}'}(H,K)$.

The following lemma describes the result of the bilinear map $- \cdot_H -$ in (6) on two standard basis elements of $\tilde{B}^{\mathcal{S}}(G,H)$ and $\tilde{B}^{\mathcal{S}}(H,K)$, respectively.

4.5 Lemma Let $G, H, K \in \mathcal{D}$, and let $(U, \alpha, V) \in E_{G,H}^{\mathcal{S}}$ and $(V', \beta, W) \in E_{H,K}^{\mathcal{S}}$.

(a) If V and V' are not H-conjugate then $[U, \alpha, V]_{G \times H}^+ \cdot_H [V', \beta, W]_{H \times K}^+ = 0$.

(b) If $V = V'$ then the following equation holds in $\mathbb{Q}A^{\mathcal{S}}(G,H)$:

$$[U, \alpha, V]_{G \times H}^+ \cdot_H [V, \beta, W]_{H \times K}^+ = \sum_{(g,h,k) \in \mathcal{A} \times \mathcal{B} \times \mathcal{C}} {}^{(g,k)}(U, \alpha c_h^{-1}\beta, W) .$$

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Here, $\mathcal{A} \subseteq G$ is a transversal for $G/p_1(N_{G \times H}(\lhd(U, \alpha, V)))$, $\mathcal{B} \subseteq N_H(V)$ is a transversal for $N_H(V)/C_H(\ker(\alpha), V)$, and $\mathcal{C} \subseteq K$ is a transversal for $K/p_2(N_{H \times K}(\lhd(V, \beta, W)))$. The right-hand side of the above equation does not depend on the choice of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

(c) One has $[U, \alpha, V]^+_{G \times H} \cdot_H [V, \beta, W]^+_{H \times K} \in \tilde{B}^{\mathcal{S}}(G, K)$.

(d) The free abelian group $\tilde{B}^{\mathcal{S}}(G, G)$ is a $\mathbb{Z}$-order in the $\mathbb{Q}$-algebra $\mathbb{Q}A^{\mathcal{S}}(G, G)^{G \times G}$, with identity element

$$\sum_{U \in \tilde{\Sigma}_G} [U, \mathrm{id}_U, U]^+_{G \times G},$$

where $\tilde{\Sigma}_G \subseteq \Sigma_G$ is a transversal of the conjugacy classes of subgroups of G.

Proof (a) This follows immediately from the definition of $- \cdot_H -$ in (7).

(b) It is straightforward to verify that the right-hand side of the equation does not depend on the choice of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. In order to prove the equation, let $\mathcal{B}_1 \subseteq H$ be a transversal for $H/C_H(\ker(\alpha), V)$ and let $\mathcal{B}_2 \subseteq H$ be a transversal for $H/C_H(V)$. Then, by Proposition 1.7, one has

$$[U, \alpha, V]^+_{G \times H} = \sum_{(g, h_1) \in \mathcal{A} \times \mathcal{B}_1} g(U, \alpha, V) h_1^{-1} \quad \text{and} \quad [V, \beta, W]^+_{H \times K} = \sum_{(h_2, k) \in \mathcal{B}_2 \times \mathcal{C}} h_2(V, \beta, W) k^{-1}.$$

Thus, by (7) and (9), we obtain

$$[U, \alpha, V]^+_{G \times H} \cdot_H [V, \beta, W]^+_{H \times K} = \sum_{(g, k) \in \mathcal{A} \times \mathcal{C}} g\Big(\sum_{\substack{(h_1, h_2) \in \mathcal{B}_1 \times \mathcal{B}_2 \\ h_1^{-1} h_2 \in N_H(V)}} \frac{|C_H(V)|}{|H|} (U, \alpha c_{h_1^{-1} h_2} \beta, W) \Big) k^{-1}.$$

Now let $\mathcal{B}_3 \subseteq H$ be a transversal of $H/N_H(V)$, let $\mathcal{B}'_1 \subseteq N_H(V)$ be a transversal of $N_H(V)/C_H(\ker(\alpha), V)$ and let $\mathcal{B}'_2 \subseteq N_H(V)$ be a transversal of $N_H(V)/C_H(V)$. Then, using the relations $h_1 = hx$ and $h_2 = hy$, we can rewrite the inner sum as

$$\sum_{h \in \mathcal{B}_3} \sum_{(x, y) \in \mathcal{B}'_1 \times \mathcal{B}'_2} \frac{|C_H(V)|}{|H|} (U, \alpha c_{x^{-1} y} \beta, W) = \frac{|C_H(V)|}{|N_H(V)|} \sum_{(x, y) \in \mathcal{B}'_1 \times \mathcal{B}'_2} (U, \alpha c_{x^{-1} y} \beta, W).$$

Finally, since for every $y \in \mathcal{B}'_2$ the elements $y^{-1}x$, $x \in \mathcal{B}'_1$, form again a transversal of $N_H(V)/C_H(\ker(\alpha), V)$ and since $C_H(\ker(\alpha), V)$ is equal to the set of all $h \in N_H(V)$ such that $\alpha c_h = \alpha$, we obtain

$$\sum_{(x, y) \in \mathcal{B}'_1 \times \mathcal{B}'_2} (U, \alpha c_{x^{-1} y} \beta, W) = [N_H(V) : C_H(V)] \sum_{x \in \mathcal{B}'_1} (U, \alpha c_x^{-1} \beta, W).$$

Altogether, we obtain the desired equation with $\mathcal{B}'_1 = \mathcal{B}$.

(c) To see that $[U, \alpha, V]^+_{G \times H} \cdot_H [V, \beta, W]^+_{H \times K} \in \tilde{B}^{\mathcal{S}}(G, K)$, recall that $\tilde{B}^{\mathcal{S}}(G, K) = A^{\mathcal{S}}(G, K)^{G \times K}$. Since the right-hand side of the equation in (b) is contained in $A^{\mathcal{S}}(G, K)$, it suffices to show that it is a $G \times K$-fixed point. But, for every $(x, y) \in G \times K$, the set $x\mathcal{A}$ (respectively $y\mathcal{C}$) is again a transversal for $G/p_1(N_{G \times H}(\lhd(U, \alpha, V)))$ (respectively $K/p_2(N_{H \times K}(\lhd(V, \beta, W)))$).

(d) An easy computation, using Proposition 1.7 again, shows that the given element is the identity element. This completes the proof. $\square$

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4.6 Let $G, H, K \in \mathcal{D}$ and assume that $\mathcal{S}_{G,H} \subseteq \Delta_{G,H}$, that $\mathcal{S}_{H,K} \subseteq \Delta_{H,K}$, and that $\mathcal{S}_{G,H}$ and $\mathcal{S}_{H,K}$ also satisfy the symmetry condition (v) in Hypothesis 1.13. Assume further that R is a commutative ring such that $|G|$, $|H|$, and $|K|$ are invertible in R. Then the R-module homomorphism given by

$$-^\circ : RA^{\mathcal{S}}(G,H) \to RA^{\mathcal{S}}(H,G), \quad (U, \alpha, V) \mapsto \frac{|C_H(V)|}{|C_G(U)|}(V, \alpha^{-1}, U),$$

is an isomorphism satisfying

$$(a^\circ)^\circ = a \quad \text{and} \quad (a \cdot_H b)^\circ = b^\circ \cdot_H a^\circ,$$

for all $a \in RA^{\mathcal{S}}(G,H)$ and all $b \in RA^{\mathcal{S}}(H,K)$. It restricts to an R-module isomorphism

$$-^\circ : R\tilde{B}^{\mathcal{S}}(G,H) \to R\tilde{B}^{\mathcal{S}}(H,G), \quad [U, \alpha, V]_{G \times H}^+ \mapsto \frac{|C_H(V)|}{|C_G(U)|}[V, \alpha^{-1}, U]_{H \times G}^+.$$

We are now ready to prove the main theorem of this section.

4.7 Theorem *Let $G, H, K \in \mathcal{D}$ and let R be a commutative ring.*

(a) *For $a \in \tilde{B}^{\mathcal{S}}(G,H)$ and $b \in \tilde{B}^{\mathcal{S}}(H,K)$, one has*

$$\rho_{G,K}^{\mathcal{S}}(a \cdot_H b) = \rho_{G,H}^{\mathcal{S}}(a) \cdot_H \rho_{H,K}^{\mathcal{S}}(b).$$

(b) *The map $\rho_{G,H}^{\mathcal{S}} : B^{\mathcal{S}}(G,H) \to \tilde{B}^{\mathcal{S}}(G,H)$ is an injective group homomorphism with finite cokernel whose order divides a power of $|G \times H|$. If $|G \times H|$ is a unit in R then the induced R-module homomorphism is an isomorphism.*

(c) *The map $\rho_{G,G}^{\mathcal{S}} : B^{\mathcal{S}}(G,G) \to \tilde{B}^{\mathcal{S}}(G,G)$ is an injective ring homomorphism with image of finite index. If $|G|$ is a unit in R then the induced R-algebra homomorphism $\rho_{G,G}^{\mathcal{S}} : RB^{\mathcal{S}}(G,G) \to R\tilde{B}^{\mathcal{S}}(G,G)$ is an isomorphism.*

(d) *Assume that $\mathcal{S}_{G,H} \leqslant \Delta_{G,H}$, that $\mathcal{S}_{G,H}$ satisfies the symmetry condition (v) in Hypothesis 1.13, and that $|G \times H|$ is invertible in R. Then*

$$\rho_{H,G}^{\mathcal{S}}(a^\circ) = \rho_{G,H}^{\mathcal{S}}(a)^\circ$$

for all $a \in RB^{\mathcal{S}}(G,H)$.

Proof (a) We may assume that $a = [X]$ and $b = [Y]$, for a (G,H)-biset X and an (H,K)-biset Y such that $\text{stab}_{G \times H}(x) \in \mathcal{S}_{G,H}$ for every $x \in X$, and $\text{stab}_{H \times K}(y) \in \mathcal{S}_{H,K}$ for every $y \in Y$. Then

$$\rho_{G,H}^{\mathcal{S}}(a) \cdot_H \rho_{H,K}^{\mathcal{S}}(b)$$

$$= \left(\sum_{(U,\alpha,V) \in E_{G,H}^{\mathcal{S}}} \frac{|X^{\lhd(U,\alpha,V)}|}{|C_G(U)|}(U, \alpha, V) \right) \cdot_H \left(\sum_{(V',\beta,W) \in E_{H,K}^{\mathcal{S}}} \frac{|Y^{\lhd(V',\beta,W)}|}{|C_H(V')|}(V', \beta, W) \right)$$

$$= \sum_{\substack{(U,\alpha,V) \in E_{G,H}^{\mathcal{S}} \\ (V,\beta,W) \in E_{H,K}^{\mathcal{S}}}} \frac{|X^{\lhd(U,\alpha,V)}| \cdot |Y^{\lhd(V,\beta,W)}|}{|C_G(U)| \cdot |H|}(U, \alpha\beta, W)$$

$$= \sum_{\substack{(U,\alpha,V) \in E_{G,H} \\ (V,\beta,W) \in E_{H,K}}} \frac{|X^{\lhd(U,\alpha,V)}| \cdot |Y^{\lhd(V,\beta,W)}|}{|C_G(U)| \cdot |H|}(U, \alpha\beta, W),$$

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since $|X^L| = 0 = |Y^M|$ for all $L \in \vartriangleleft_{G,H} \smallsetminus \mathcal{S}_{G,H}$ and all $M \in \vartriangleleft_{H,K} \smallsetminus \mathcal{S}_{H,K}$.

We fix a standard basis element $(U, \gamma, W) \in E^{\mathcal{S}}_{G,K}$ of $A^{\mathcal{S}}(G, K)$. By the above, the coefficient of $\rho^{\mathcal{S}}_{G,H}(a) \cdot_H \rho^{\mathcal{S}}_{H,K}(b)$ at the basis element (U, γ, W) is equal to the number

$$\frac{1}{|C_G(U)| \cdot |H|} \sum_{(\alpha, V, \beta) \in \Gamma_H(U, \gamma, W)} |X^{\vartriangleleft(U, \alpha, V)}| \cdot |Y^{\vartriangleleft(V, \beta, W)}|,$$

where $\Gamma_H(U, \gamma, W)$ is defined as in Subsection 2.2(b). On the other hand, the coefficient of $\rho^{\mathcal{S}}_{G,K}(a \cdot_H b) = \rho^{\mathcal{S}}_{G,K}([X \times_H Y])$ at the basis element (U, γ, W) is equal to the number

$$\frac{|(X \times_H Y)^{\vartriangleleft(U, \gamma, W)}|}{|C_G(U)|}.$$

But, by Theorem 2.3, these two numbers are equal.

(b) Using the bijection (5), we see that the standard bases of $B^{\mathcal{S}}(G, H)$ and $\tilde{B}^{\mathcal{S}}(G, H)$ are both parametrized by the set of $G \times H$-conjugacy classes of $\mathcal{S}_{G,H}$. Arranging basis elements with respect to ascending group order, the matrix describing $\rho^{\mathcal{S}}_{G,H}$ with respect to these bases is upper triangular with diagonal entries of the form $[N_{G \times H}(L) : L]/|C_G(U)|$, where $L = \vartriangleleft(U, \alpha, V)$, cf. the proof of Proposition 1.14(c). The statements in Part (b) follow easily.

(c) This is an immediate consequence of Parts (a) and (b). Note that $\tilde{B}^{\mathcal{S}}(G, G)$ is a ring, by Lemma 4.5(d).

(d) We may assume that $a = [X]$, for a bifree (G, H)-biset X such that $\mathrm{stab}_{G \times H}(x) \in \mathcal{S}_{G,H}$ for every $x \in X$. Then we have

$$\begin{aligned}
\rho^{\mathcal{S}}_{G,H}([X])^{\circ} &= \Big[\sum_{(U, \alpha, V) \in I^{\mathcal{S}}_{G,H}} \frac{|X^{\Delta(U, \alpha, V)}|}{|C_G(U)|} (U, \alpha, V) \Big]^{\circ} \\
&= \sum_{(U, \alpha, V) \in I^{\mathcal{S}}_{G,H}} \frac{|X^{\Delta(U, \alpha, V)}|}{|C_H(V)|} (V, \alpha^{-1}, U) \\
&= \sum_{(U, \alpha, V) \in I^{\mathcal{S}}_{G,H}} \frac{|(X^{\circ})^{\Delta(V, \alpha^{-1}, U)}|}{|C_H(V)|} (V, \alpha^{-1}, U) \\
&= \rho^{\mathcal{S}}_{H,G}([X]^{\circ}).
\end{aligned}$$

Here we used that the map $I^{\mathcal{S}}_{G,H} \to I^{\mathcal{S}}_{H,G}$, $(U, \alpha, V) \mapsto (V, \alpha^{-1}, U)$, is a bijection by Hypothesis 1.13(v), and that $x \in X^L$ if and only if $x^{\circ} \in (X^{\circ})^{L^{\circ}}$, for all $x \in X$ and all $L \leqslant G \times H$. $\qquad \square$

4.8 Remark The inverse of the isomorphism $\rho^{\vartriangleleft}_{G,H} \colon \mathbb{Q}B^{\vartriangleleft}(G, H) \to \mathbb{Q}\tilde{B}^{\vartriangleleft}(G, H)$ can be given explicitly. In fact, one can use the inversion formula in [Gl] to see that

$$(\rho^{\vartriangleleft}_{G,H})^{-1}([U, \alpha, V]^+_{G \times H}) = \frac{|C_G(U)|}{|N_{G \times H}(\vartriangleleft(U, \alpha, V))|} \sum_{W \leqslant V} |W| \cdot \mu(W, V) \cdot [G \times H / \vartriangleleft(\alpha(W), \alpha, W)].$$

Here, we used that the map $W \mapsto (\alpha(W), \alpha|_W, W)$ defines an isomorphism between the partially ordered sets of subgroups of V and subgroups of $\vartriangleleft(U, \alpha, V)$. Also, μ denotes the Möbius function of the partially ordered set of subgroups of H.

4.9 Remark Given a set $\mathcal{D}$, and $\mathcal{S}_{G,H} \subseteq \lhd_{G,H}$ for $G, H \in \mathcal{D}$, such that Condition (I) in Hypothesis 1.13 is satisfied, we can define a ring $\tilde{A}^{\mathcal{D},\mathcal{S}}$ by

$$\tilde{A}^{\mathcal{D},\mathcal{S}} := \bigoplus_{G,H \in \mathcal{D}} \tilde{B}^{\mathcal{S}}(G,H) \,,$$

with multiplication defined as follows: for $a \in \tilde{B}^{\mathcal{S}}(G,H)$ and $b \in \tilde{B}^{\mathcal{S}}(H',K)$ set $a \cdot b := a \cdot_H b$ if $H = H'$, and $a \cdot b := 0$ if $H \neq H'$. By Theorem 4.7, the collection of the maps $\rho_{G,H}^{\mathcal{S}} \colon B^{\mathcal{S}}(G,H) \to \tilde{B}^{\mathcal{S}}(G,H)$, $G, H \in \mathcal{D}$, defines an injective ring homomorphism

$$\rho \colon A^{\mathcal{D},\mathcal{S}} \to \tilde{A}^{\mathcal{D},\mathcal{S}} \,.$$

Here, $A^{\mathcal{D},\mathcal{S}}$ is as in 1.16. For a commutative ring R, we set $\tilde{A}_R^{\mathcal{D},\mathcal{S}} := R \otimes_{\mathbb{Z}} \tilde{A}^{\mathcal{D},\mathcal{S}}$ and identify $\tilde{A}_R^{\mathcal{D},\mathcal{S}}$ canonically with $\bigoplus_{G,H \in \mathcal{D}} R\tilde{B}^{\mathcal{S}}(G,H)$. If, for every $G \in \mathcal{D}$, the number $|G|$ is a unit in R then the above ring homomorphism induces an R-algebra isomorphism

$$\rho \colon A_R^{\mathcal{D},\mathcal{S}} \xrightarrow{\sim} \tilde{A}_R^{\mathcal{D},\mathcal{S}} \,.$$

In this case, the abelian categories $_{A_R^{\mathcal{D},\mathcal{S}}}\mathsf{Mod}^*$ and $_{\tilde{A}_R^{\mathcal{D},\mathcal{S}}}\mathsf{Mod}^*$ are isomorphic via ρ. Here, the category $_{\tilde{A}_R^{\mathcal{D},\mathcal{S}}}\mathsf{Mod}^*$ ist defined in analogy to the category $_{A_R^{\mathcal{D},\mathcal{S}}}\mathsf{Mod}^*$ in 1.16. Combining this with the category equivalence in 1.16(b) we obtain a category equivalence between the biset functor category $\mathsf{Func}_R^{\mathcal{D},\mathcal{S}}$ and the category $_{\tilde{A}_R^{\mathcal{D},\mathcal{S}}}\mathsf{Mod}^*$. This is useful, since the ring structure of $\tilde{A}_R^{\mathcal{D},\mathcal{S}}$ is much more transparent than the ring structure of $A_R^{\mathcal{D},\mathcal{S}}$, as we will see in the following sections.

5 The multiplicative structure of $\tilde{B}^{\Delta}(G,G)$

In this section we introduce a natural direct-product decomposition of the ghost ring $\tilde{B}^{\Delta}(G,G)$ into subrings indexed by the isomorphism classes of subgroups of G. We also give two descriptions of these components in terms of Hecke algebras, i.e., endomorphism rings of permutation modules.

Throughout this section let $\mathcal{T}$ denote a set of representatives of the isomorphism classes of finite groups. Recall that, for a finite group G and $T \in \mathcal{T}$, we denote by $\Sigma_G(T)$ the set of subgroups of G that are isomorphic to T. We assume again that $\mathcal{D}$ is a class of finite groups and that, for every $G, H \in \mathcal{D}$, $\mathcal{S}_{G,H} \subseteq \Delta_{G,H}$ is a set of subgroups of $G \times H$ satisfying Condition (I) in Hypothesis 1.13. We emphasize that, in this section, we assume that $\mathcal{S}_{G,H}$ is contained in $\Delta_{G,H}$. The constructions we present will not work for the larger ghost ring $\tilde{B}^{\lhd}(G,G)$ of the left-free double Burnside ring.

Again, R will denote a commutative ring.

5.1 *A decomposition of $\tilde{B}^{\Delta}(G,H)$.* Let $G, H, K \in \mathcal{D}$. Recall from 4.1 that $I_{G,H}^{\mathcal{S}}$ consists of all triples $(U, \alpha, V) \in I_{G,H}$ such that $\Delta(U, \alpha, V) \in \mathcal{S}_{G,H}$. The set $I_{G,H}^{\mathcal{S}}$ decomposes into a disjoint union

$$I_{G,H}^{\mathcal{S}} = \bigcup_{T \in \mathcal{T}} I_{G,H,T}^{\mathcal{S}} \,,$$

where

$$I_{G,H,T}^{\mathcal{S}} := \{(U, \alpha, V) \in I_{G,H} \mid U \cong T \cong V\} \,.$$

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This decomposition gives rise to direct-sum decompositions

$$A^{\mathcal{S}}(G,H) = \bigoplus_{T \in \mathcal{T}} A_T^{\mathcal{S}}(G,H) \quad \text{and} \quad RA^{\mathcal{S}}(G,H) = \bigoplus_{T \in \mathcal{T}} RA_T^{\mathcal{S}}(G,H) \,,$$

where $A_T^{\mathcal{S}}(G,H)$ (respectively $RA_T^{\mathcal{S}}(G,H)$) is the $\mathbb{Z}$-span (respectively R-span) of the subset $I_{G,H,T}^{\mathcal{S}}$ of $I_{G,H}^{\mathcal{S}}$. Note that in the above direct sum all but finitely many summands are equal to $\{0\}$, since $I_{G,H,T}^{\mathcal{S}}$ is non-empty only if G and H have subgroups that are isomorphic to T. Note further that $I_{G,H,T}^{\mathcal{S}}$ is a $G \times H$-stable subset of $I_{G,H}^{\mathcal{S}}$, for all $T \in \mathcal{T}$. Therefore, taking $G \times H$-fixed points, we obtain decompositions

$$\tilde{B}^{\mathcal{S}}(G,H) = \bigoplus_{T \in \mathcal{T}} \tilde{B}_T^{\mathcal{S}}(G,H) \quad \text{and} \quad R\tilde{B}^{\mathcal{S}}(G,H) = \bigoplus_{T \in \mathcal{T}} R\tilde{B}_T^{\mathcal{S}}(G,H) \,, \tag{11}$$

where $\tilde{B}_T^{\mathcal{S}}(G,H) := A_T^{\mathcal{S}}(G,H)^{G \times H}$ for $T \in \mathcal{T}$. By the definition of the multiplication in $A^{\Delta}(G,H)$, cf. (7), the decomposition (11) satisfies

$$R\tilde{B}_{T_1}^{\mathcal{S}}(G,H) \cdot_H R\tilde{B}_{T_2}^{\mathcal{S}}(H,K) = 0 \quad \text{if } T_1 \neq T_2. \tag{12}$$

Thus, the bilinear map $- \cdot_H -$ is the collection of componentwise bilinear maps with respect to the decomposition in (11).

We can now write the mark homomorphism as a collection

$$\rho_{G,H}^{\mathcal{S}} = (\rho_{G,H,T}^{\mathcal{S}})_{T \in \mathcal{T}} \colon B^{\mathcal{S}}(G,H) \xrightarrow{\sim} \bigoplus_{T \in \mathcal{T}} \tilde{B}_T^{\mathcal{S}}(G,H) \tag{13}$$

of homomorphisms $\rho_{G,H,T}^{\mathcal{S}} \colon B^{\mathcal{S}}(G,H) \to \tilde{B}_T^{\mathcal{S}}(G,H)$.

If $\mathcal{S}_{G,H} = \Delta_{G,H}$ then we will use the notation $A_T^{\Delta}(G,H)$ and $\tilde{B}_T^{\Delta}(G,H)$ for $A_T^{\mathcal{S}}(G,H)$ and $\tilde{B}_T^{\mathcal{S}}(G,H)$, respectively.

Combining the above statements with those from Lemma 4.5 and Theorem 4.7, we obtain the following theorem.

5.2 Theorem *Let $G, H, K \in \mathcal{D}$.*
(a) *Let $a = (a_T) \in R\tilde{B}^{\mathcal{S}}(G,H)$, and $b = (b_T) \in R\tilde{B}^{\mathcal{S}}(H,K)$. Then one has*

$$(a_T) \cdot_H (b_T) = (a_T \cdot_H b_T).$$

(b) *For $T \in \mathcal{T}$, the R-module $R\tilde{B}_T^{\mathcal{S}}(G,G)$ is an R-subalgebra of $RA_T^{\mathcal{S}}(G,G)$ with identity element*

$$e_{G,T} := \sum_{U \in \tilde{\Sigma}_G(T)} [U, \mathrm{id}_U, U]_{G \times G}^+ \,,$$

where $\tilde{\Sigma}_G(T) \subseteq \Sigma_G(T)$ denotes a set of representatives of the G-conjugacy classes of $\Sigma_G(T)$. If $|G|$ is invertible in R then

$$\rho_{G,G}^{\mathcal{S}} \colon RB^{\mathcal{S}}(G,G) \xrightarrow{\sim} \bigoplus_{T \in \mathcal{T}} R\tilde{B}_T^{\mathcal{S}}(G,G) \,.$$

is an isomorphism of R-algebras.

Next, we will give an alternative construction of ghost groups and mark homomorphisms for $B^{\mathcal{S}}(G,H) \subseteq B^{\Delta}(G,H)$ in terms of homomorphism groups between permutation modules.

5.3 *The mark homomorphism $\sigma_{G,H}^{\mathcal{S}}$.* (a) For $T \in \mathcal{T}$, let $\mathrm{Inj}(T,G)$ denote the set of injective homomorphisms $\lambda \colon T \to G$. Note that $\mathrm{Inj}(T,G)$ is a $(G, \mathrm{Aut}(T))$-biset via $g\lambda\omega := c_g \circ \lambda \circ \omega$ for $g \in G$, $\lambda \in \mathrm{Inj}(T,G)$ and $\omega \in \mathrm{Aut}(T)$. We denote by $\overline{\mathrm{Inj}}(T,G)$ the set of G-orbits of $\mathrm{Inj}(T,G)$, and denote by $[\lambda]$ the G-orbit of an element $\lambda \in \mathrm{Inj}(T,G)$. The set $\overline{\mathrm{Inj}}(T,G)$ is still a right $\mathrm{Aut}(T)$-set and the group $\mathrm{Inn}(T)$ of inner automorphisms of T acts trivially on $\overline{\mathrm{Inj}}(T,G)$, since $\lambda \circ c_t = c_{\lambda(t)} \circ \lambda$ for $t \in T$ and $\lambda \in \mathrm{Inj}(T,G)$. Thus, we may consider $\overline{\mathrm{Inj}}(T,G)$ as a right $\mathrm{Out}(T)$-set, where $\mathrm{Out}(T) := \mathrm{Aut}(T)/\mathrm{Inn}(T)$ denotes the group of outer automorphisms of T. Note that, of course, $\mathrm{Inj}(T,G)$ is empty if T is not isomorphic to a subgroup of G.

(b) For $T \in \mathcal{T}$ consider the map

$$f \colon \mathrm{Inj}(T,G) \times \mathrm{Inj}(T,H) \to I_{G,H,T}, \quad (\lambda, \mu) \mapsto \Delta(\lambda(T), \lambda\mu^{-1}, \mu(T)).$$

This map is clearly surjective, and satisfies $f(c_g \lambda\omega, c_h \mu\omega) = {}^{(g,h)}f(\lambda, \mu)$ for all $g \in G$, $h \in H$ and $\omega \in \mathrm{Aut}(T)$. Thus, we obtain an induced $G \times H$-equivariant surjective map

$$\mathrm{Inj}(T,G) \times_{\mathrm{Aut}(T)} \mathrm{Inj}(T,H) \to I_{G,H,T}.$$

Strictly speaking we should write $\mathrm{Inj}(T,H)^{\circ}$ in order to view $\mathrm{Inj}(T,H)$ as a left $\mathrm{Aut}(T)$-set, but we prefer to keep the notation simple. It is straightforward to see that this map is also injective. Consequently it induces a bijection

$$\overline{\mathrm{Inj}}(T,G) \times_{\mathrm{Aut}(T)} \overline{\mathrm{Inj}}(T,H) \xrightarrow{\sim} I_{G,H,T}/(G \times H). \tag{14}$$

Recall from (2) that $I_{G,H,T}$ is also in $G \times H$-equivariant bijection with the set of twisted diagonal subgroups of $G \times H$ that are isomorphic to T.

(c) For $T \in \mathcal{T}$ and finite groups G and H, we define the $\mathbb{Z}$-linear map

$$\sigma_{G,H,T} \colon B^{\Delta}(G,H) \to \mathrm{Hom}_{\mathbb{Z}\mathrm{Out}(T)}(\mathbb{Z}\overline{\mathrm{Inj}}(T,H), \mathbb{Z}\overline{\mathrm{Inj}}(T,G)),$$

$$[X] \mapsto \left([\mu] \mapsto \sum_{[\lambda] \in \overline{\mathrm{Inj}}(T,G)} \frac{|X^{\Delta(\lambda(T), \lambda\mu^{-1}, \mu(T))}|}{|C_G(\lambda(T))|} [\lambda] \right),$$

where X is any bifree (G,H)-biset. Note that the integer $|X^{\Delta(\lambda(T), \lambda\mu^{-1}, \mu(T))}|$ is divisible by $|C_G(U)|$, since X is left-free. Since the map $\sigma_{G,H,T}([X])$ is defined on a $\mathbb{Z}$-basis, and since the definition does not depend on the choices of λ and μ in their classes, it is a well-defined group homomorphism. Moreover it is easy to verify that it respects the $\mathbb{Z}\mathrm{Aut}(T)$-module structures. Collecting all these maps we obtain a map

$$\sigma_{G,H} \colon B^{\Delta}(G,H) \to \bigoplus_{T \in \mathcal{T}} \mathrm{Hom}_{\mathbb{Z}\mathrm{Out}(T)}(\mathbb{Z}\overline{\mathrm{Inj}}(T,H), \mathbb{Z}\overline{\mathrm{Inj}}(T,G)). \tag{15}$$

(d) Assume that $G, H \in \mathcal{D}$. Then the map $\sigma_{G,H,T}$ in Part (c) restricts to a map

$$\sigma_{G,H,T}^{\mathcal{S}} \colon B^{\mathcal{S}}(G,H) \to \mathrm{Hom}_{\mathbb{Z}\mathrm{Out}(T)}^{\mathcal{S}}(\mathbb{Z}\overline{\mathrm{Inj}}(T,H), \mathbb{Z}\overline{\mathrm{Inj}}(T,G)), \tag{16}$$

where the latter set consists of those $\mathbb{Z}\mathrm{Out}(T)$-module homomorphisms $f \colon \mathbb{Z}\overline{\mathrm{Inj}}(T,H) \to \mathbb{Z}\overline{\mathrm{Inj}}(T,G)$ with the property that, for every $\mu \in \mathrm{Inj}(T,H)$, the element $f([\mu])$ lies in the $\mathbb{Z}$-span of those standard

basis elements $[\lambda]$ satisfying $(\lambda(T), \lambda\mu^{-1}, \mu(T)) \in I_{G,H}^{\mathcal{S}}$. In fact, since $\mathcal{S}_{G,H}$ is closed under $G \times H$-conjugation and under taking subgroups, we have $X^{\Delta(\lambda(T), \lambda\mu^{-1}, \mu(T))} = \emptyset$ for $[X] \in B^{\mathcal{S}}(G,H)$, unless $\Delta(\lambda(T), \lambda\mu^{-1}, \mu(T)) \in \mathcal{S}_{G,H}$. Moreover, the map $\sigma_{G,H}$ in (15) restricts to a map

$$\sigma_{G,H}^{\mathcal{S}} = (\sigma_{G,H,T}^{\mathcal{S}})_{T \in \mathcal{T}} \colon B^{\mathcal{S}}(G,H) \to \bigoplus_{T \in \mathcal{T}} \mathrm{Hom}_{\mathbb{Z}\mathrm{Out}(T)}^{\mathcal{S}}(\mathbb{Z}\overline{\mathrm{Inj}}(T,H), \mathbb{Z}\overline{\mathrm{Inj}}(T,G)) \,. \tag{17}$$

Finally, tensoring the above maps with R over $\mathbb{Z}$ yields an R-module homomorphism

$$\sigma_{G,H}^{\mathcal{S}} = (\sigma_{G,H,T}^{\mathcal{S}})_{T \in \mathcal{T}} \colon RB^{\mathcal{S}}(G,H) \to \bigoplus_{T \in \mathcal{T}} \mathrm{Hom}_{R\mathrm{Out}(T)}^{\mathcal{S}}(R\overline{\mathrm{Inj}}(T,H), R\overline{\mathrm{Inj}}(T,G)) \,. \tag{18}$$

In fact, since $\mathbb{Z}\overline{\mathrm{Inj}}(T,H)$ and $\mathbb{Z}\overline{\mathrm{Inj}}(T,G)$ are permutation modules, the canonical map between the tensor product of R with the homomorphism group in (16) to the latter homomorphism module is an isomorphism.

Note also that all but finitely many summands are trivial in all the above direct sums running over $T \in \mathcal{T}$.

5.4 *The connecting map $\tau_{G,H,T}^{\mathcal{S}}$.* Let $G, H \in \mathcal{D}$. We define the group homomorphism

$$\tau_{G,H,T}^{\mathcal{S}} \colon \tilde{B}_T^{\mathcal{S}}(G,H) \to \mathrm{Hom}_{\mathbb{Z}\mathrm{Out}(T)}^{\mathcal{S}}(\mathbb{Z}\overline{\mathrm{Inj}}(T,H), \mathbb{Z}\overline{\mathrm{Inj}}(T,G)) \,, \tag{19}$$

by setting

$$(\tau_{G,H,T}^{\mathcal{S}}(a))([\mu]) := \sum_{[\lambda] \in \overline{\mathrm{Inj}}(T,G)} a_{(\lambda(T), \lambda\mu^{-1}, \mu(T))} \, [\lambda] \,,$$

for $a = \sum_{(U,\alpha,V) \in I_{G,H,T}^{\mathcal{S}}} a_{(U,\alpha,V)}(U,\alpha,V) \in \tilde{B}_T^{\mathcal{S}}(G,H)$ and $[\mu] \in \overline{\mathrm{Inj}}(T,H)$. It is straightforward to check that this map is well defined.

The following theorem shows that the direct sum of homomorphism groups in (17) can serve as an alternative ghost group, that $\sigma_{G,H}$ can serve as a mark homomorphism translating the tensor product construction on bisets into componentwise composition of homomorphisms, and that the map $\tau_{G,H}^{\mathcal{S}}$ is an isomorphism that translates between these two constructions.

A special case of Part (c) of the following theorem can be derived from [Bc96b, Théorème 2] or [Bc96b, Proposition 7], using the statement from the second paragraph on page 753 in [Bc96b] and translating our setting via duality (cf. 1.4) to the realm of right-free bisets.

5.5 Theorem *Let $G, H, K \in \mathcal{D}$ and let R be a commutative ring.*

(a) *For $T \in \mathcal{T}$, $a \in B^{\mathcal{S}}(G,H)$ and $b \in B^{\mathcal{S}}(H,K)$, one has*

$$\sigma_{G,K,T}^{\mathcal{S}}(a \cdot_H b) = \sigma_{G,H,T}^{\mathcal{S}}(a) \circ \sigma_{H,K,T}^{\mathcal{S}}(b) \,.$$

(b) *The group homomorphism $\sigma_{G,H}^{\mathcal{S}}$ in (17) is injective with finite cokernel. If $|G \times H|$ is invertible in R then the induced R-module homomorphism in (18) is an isomorphism.*

(c) *The map*

$$\sigma_{G,G}^{\mathcal{S}} \colon B^{\mathcal{S}}(G,G) \to \prod_{T \in \mathcal{T}} \mathrm{End}_{\mathbb{Z}\mathrm{Out}(T)}^{\mathcal{S}}(\mathbb{Z}\overline{\mathrm{Inj}}(T,G))$$

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is an injective ring homomorphism with image of finite index, where the multiplication in the codomain is given by componentwise composition. If $|G|$ is invertible in R then the induced map

$$\sigma^{\mathcal{S}}_{G,G}\colon RB^{\mathcal{S}}(G,G) \to \prod_{T\in\mathcal{T}} \mathrm{End}^{\mathcal{S}}_{R\mathrm{Out}(T)}(R\overline{\mathrm{Inj}}(T,G))$$

is an R-algebra isomorphism.

(d) For $T \in \mathcal{T}$, the map $\tau^{\mathcal{S}}_{G,H,T}$ is a group isomorphism and it satisfies

$$\tau^{\mathcal{S}}_{G,H,T} \circ \rho^{\mathcal{S}}_{G,H,T} = \sigma^{\mathcal{S}}_{G,H,T}\,.$$

In particular, the diagram

$$B^{\mathcal{S}}(G,H)$$

$$\rho^{\mathcal{S}}_{G,H} \qquad\qquad \sigma^{\mathcal{S}}_{G,H}$$

$$\bigoplus_{T\in\mathcal{T}} \tilde{B}^{\mathcal{S}}(G,H) \quad\xrightarrow{\;(\tau^{\mathcal{S}}_{G,H,T})\;}\quad \bigoplus_{T\in\mathcal{T}} \mathrm{Hom}^{\mathcal{S}}_{\mathbb{Z}\mathrm{Out}(T)}(\mathbb{Z}\overline{\mathrm{Inj}}(T,H), \mathbb{Z}\overline{\mathrm{Inj}}(T,G))$$

is commutative. Moreover, for $T \in \mathcal{T}$, $a \in \tilde{B}^{\mathcal{S}}_T(G,H)$, and $b \in \tilde{B}^{\mathcal{S}}_T(H,K)$, one has

$$\tau^{\mathcal{S}}_{G,K,T}(a \cdot_H b) = \tau^{\mathcal{S}}_{G,H,T}(a) \circ \tau^{\mathcal{S}}_{H,K,T}(b)\,.$$

Proof (a) We may assume that $a = [X]$ and $b = [Y]$ for bifree bisets X and Y. Let $\nu \in \mathrm{Inj}(T,K)$. Then, by Theorem 2.5, we obtain

$$\sigma^{\mathcal{S}}_{G,K,T}([X] \cdot_H [Y])([\nu]) = \sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,G)} \frac{|(X \times_H Y)^{\Delta(\lambda(T),\lambda\nu^{-1},\nu(T))}|}{|C_G(\lambda(T))|}[\lambda]$$

$$= \sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,G)} \frac{1}{|H|} \sum_{(\alpha,V,\beta)\in\Gamma_H(\lambda(T),\lambda\nu^{-1},\nu(T))} \frac{|X^{\Delta(\lambda(T),\alpha,V)}| \cdot |Y^{\Delta(V,\beta,\nu(T))}|}{|C_G(\lambda(T))|}[\lambda]$$

$$= \sum_{[\lambda]\in\overline{\mathrm{Inj}}(T,G)} \sum_{[\mu]\in\overline{\mathrm{Inj}}(T,H)} \frac{|X^{\Delta(\lambda(T),\lambda\mu^{-1},\mu(T))}| \cdot |Y^{\Delta(\mu(T),\mu\nu^{-1},\nu(T))}|}{|C_G(\lambda(T))| \cdot |C_H(\mu(T))|}[\lambda]$$

$$= \sigma^{\mathcal{S}}_{G,H,T}([X])\Big(\sum_{[\mu]\in\overline{\mathrm{Inj}}(T,H)} \frac{|Y^{\Delta(\mu(T),\mu\nu^{-1},\nu(T))}|}{|C_H(\mu(T))|}[\mu]\Big) = (\sigma^{\mathcal{S}}_{G,H,T}([X]) \circ \sigma^{\mathcal{S}}_{H,K,T}([Y]))([\nu])\,.$$

Here we used that, for a fixed $\lambda \in \mathrm{Inj}(T,G)$, the map $\mu \mapsto (\lambda\mu^{-1}, \mu(T), \mu\nu^{-1})$ defines a bijection between $\mathrm{Inj}(T,H)$ and $\Gamma_H(\lambda(T), \lambda\nu^{-1}, \nu(T))$ with inverse $(\alpha,V,\beta) \mapsto \alpha^{-1}\lambda$, and that $\mathrm{stab}_H(\mu) = C_H(\mu(T))$.

(d) We prove Part (d) before we prove Parts (b) and (c). Define the map

$$\tau'^{\mathcal{S}}_{G,H,T}\colon \mathrm{Hom}^{\mathcal{S}}_{\mathbb{Z}\mathrm{Out}(T)}(\mathbb{Z}\overline{\mathrm{Inj}}(T,H), \mathbb{Z}\overline{\mathrm{Inj}}(T,G)) \to \tilde{B}^{\mathcal{S}}_T(G,H)$$

as follows: if $f \in \mathrm{Hom}^{\mathcal{S}}_{\mathbb{Z}\mathrm{Out}(T)}(\mathbb{Z}\overline{\mathrm{Inj}}(T, H), \mathbb{Z}\overline{\mathrm{Inj}}(T, G))$ is a $\mathbb{Z}\mathrm{Out}(T)$-homomorphism and if f is represented by the integral matrix $(a_{[\lambda], [\mu]})$ with respect to the standard bases $\overline{\mathrm{Inj}}(T, G)$ and $\overline{\mathrm{Inj}}(T, H)$ then we set

$$\tau'^{\mathcal{S}}_{G,H,T}(f) := \sum_{\substack{\lambda \times_{\mathrm{Aut}(T)} \mu \in \\ \mathrm{Inj}(T,G) \times_{\mathrm{Aut}(T)} \mathrm{Inj}(T,H)}} a_{[\lambda],[\mu]} \; (\lambda(T), \lambda\mu^{-1}, \mu(T)) \,.$$

Note that $a_{[\lambda],[\mu]} = 0$, unless $(\lambda(T), \lambda\mu^{-1}, \mu(T)) \in I^{\mathcal{S}}_{G,H,T}$. Also note that, since f is a $\mathbb{Z}\mathrm{Out}(T)$-module homomorphism, one has $a_{[\lambda\omega],[\mu\omega]} = a_{[\lambda],[\mu]}$, for $\omega \in \mathrm{Aut}(T)$. A straightforward verification shows that $\tau^{\mathcal{S}}_{G,H,T}$ and $\tau'^{\mathcal{S}}_{G,H,T}$ are inverses of each other. Moreover, by the very definitions of $\rho^{\mathcal{S}}_{G,H,T}$, $\sigma^{\mathcal{S}}_{G,H,T}$ and $\tau^{\mathcal{S}}_{G,H,T}$, we see that $\tau^{\mathcal{S}}_{G,H,T} \circ \rho^{\mathcal{S}}_{G,H,T} = \sigma^{\mathcal{S}}_{G,H,T}$.

For the last statement of Part (d), we apply Theorem 4.7(b) to choose $a' \in \mathbb{Q}B^{\mathcal{S}}(G, H)$ and $b' \in \mathbb{Q}B^{\mathcal{S}}(H, K)$ satisfying $\rho^{\mathcal{S}}_{G,H,T}(a') = a$ and $\rho^{\mathcal{S}}_{H,K,T}(b') = b$. Then, using the second statement of Part (d), Part (a), and Theorem 4.7(a), we have

$$\tau^{\mathcal{S}}_{G,H,T}(a) \circ \tau^{\mathcal{S}}_{H,K,T}(b) = \sigma^{\mathcal{S}}_{G,H,T}(a') \circ \sigma^{\mathcal{S}}_{H,K,T}(b') = \sigma^{\mathcal{S}}_{G,K,T}(a' \cdot_H b')$$
$$= (\tau^{\mathcal{S}}_{G,K,T} \circ \rho^{\mathcal{S}}_{G,K,T})(a' \cdot_H b') = \tau^{\mathcal{S}}_{G,K,T}(\rho^{\mathcal{S}}_{G,H,T}(a') \cdot_H \rho^{\mathcal{S}}_{H,K,T}(b')) = \tau^{\mathcal{S}}_{G,K,T}(a \cdot_H b) \,.$$

(b) This follows immediately from the corresponding statement for $\rho^{\mathcal{S}}_{G,H}$ in Theorem 4.7(b), the commutativity of the triangle diagram in Part (d), and the fact that $\tau^{\mathcal{S}}_{G,H,T}$ is an isomorphism for all $T \in \mathcal{T}$.

(c) The map $\sigma^{\mathcal{S}}_{G,G}$ is a ring homomorphism by Part (a). The remaining statements follow immediately from Part (b). $\qquad \square$

5.6 *The mark homomorphism $\tilde{\sigma}^{\mathcal{S}}_G$.* Next we consider the case where $G = H$ more closely. For $G \in \mathcal{D}$ we define a category $\mathcal{S}_G$ whose objects are the subgroups of G and where any morphism set $\mathrm{Hom}_{\mathcal{S}_G}(V, U)$ is defined as the set of all group homomorphisms $\phi \colon V \to U$ such that $\{(\phi(v), v)) \mid v \in V\} \in \mathcal{S}_{G,G}$. Note that automatically each such ϕ is injective, since $\mathcal{S}_{G,G} \subseteq \Delta_{G,G}$. The conditions in Hypothesis 1.13 imply that this is in fact a category with the usual composition of homomorphisms, and that every conjugation map $c_g \colon V \to U$, for $g \in G$, between subgroups V and U of G is a morphism in this category. We denote by $\tilde{\mathcal{S}}_G$ a set of representatives of the isomorphism classes of objects of $\mathcal{S}_G$. For $U \leqslant G$ we set $\mathrm{Aut}_{\mathcal{S}_G}(U) := \mathrm{Hom}_{\mathcal{S}_G}(U, U)$ and $\mathrm{Out}_{\mathcal{S}_G}(U) := \mathrm{Aut}_{\mathcal{S}_G}(U)/\mathrm{Inn}(U)$. Moreover we set $\overline{\mathrm{Hom}}_{\mathcal{S}_G}(U, V) := \mathrm{Inn}(V) \backslash \mathrm{Hom}_{\mathcal{S}_G}(U, V)$. This set can be considered as a right $\mathrm{Out}_{\mathcal{S}_G}(U)$-set by composition. Thus, specializing to $V = G$, we obtain a permutation $\mathbb{Z}\mathrm{Out}_{\mathcal{S}_G}(U)$-module $\mathbb{Z}\overline{\mathrm{Hom}}_{\mathcal{S}_G}(U, G)$ and its endomorphism ring

$$\mathrm{End}_{\mathbb{Z}\mathrm{Out}_{\mathcal{S}_G}(U)}(\mathbb{Z}\overline{\mathrm{Hom}}_{\mathcal{S}_G}(U, G)) \,. \tag{20}$$

Similarly as in 5.3(c), we obtain, for every $U \leqslant G$, a well-defined group homomorphism

$$\tilde{\sigma}^{\mathcal{S}}_{G,U} \colon B^{\mathcal{S}}(G, G) \to \mathrm{End}_{\mathbb{Z}\mathrm{Out}_{\mathcal{S}_G}(U)}(\mathbb{Z}\overline{\mathrm{Hom}}_{\mathcal{S}_G}(U, G)) \,,$$

$$[X] \mapsto \left([\psi] \mapsto \sum_{[\phi] \in \overline{\mathrm{Hom}}_{\mathcal{S}_G}(U,G)} \frac{|X^{\Delta(\phi(U), \phi\psi^{-1}, \psi(U))}|}{|C_G(\phi(U))|}[\phi]\right) \,.$$

The collection of these maps, for $U \in \tilde{\mathcal{S}}_G$, defines a group homomorphism

$$\tilde{\sigma}_G^{\mathcal{S}} \colon B^{\mathcal{S}}(G,G) \to \bigoplus_{U \in \tilde{\mathcal{S}}_G} \operatorname{End}_{\mathbb{Z}\operatorname{Out}_{\mathcal{S}_G}(U)}(\mathbb{Z}\overline{\operatorname{Hom}}_{\mathcal{S}_G}(U,G)) \tag{21}$$

which induces an R-module homomorphism

$$\tilde{\sigma}_G^{\mathcal{S}} \colon RB^{\mathcal{S}}(G,G) \to \bigoplus_{U \in \tilde{\mathcal{S}}_G} \operatorname{End}_{R\operatorname{Out}_{\mathcal{S}_G}(U)}(R\overline{\operatorname{Hom}}_{\mathcal{S}_G}(U,G)) \,. \tag{22}$$

5.7 Theorem *Let $G \in \mathcal{D}$ and assume the notation from 5.6.*
    (a) *The map in (21) is an injective ring homomorphism with image of finite index.*
    (b) *If $|G|$ is a unit in R then the map in (22) is an isomorphism of R-algebras.*

Proof (a) First we show that $\tilde{\sigma}_G^{\mathcal{S}}$ is injective. Assume that $a \in B^{\mathcal{S}}(G,G)$ is such that $\tilde{\sigma}_G^{\mathcal{S}}(a) = 0$. Then $\Phi_L(a) = 0$ for all $L = \Delta(\phi(U), \phi\psi^{-1}, \psi(U))$, where $U \in \tilde{\mathcal{S}}_G$, and $\phi, \psi \in \operatorname{Hom}_{\mathcal{S}_G}(U,G)$. However, it is straightforward to show that the $G \times G$-equivariant maps

$$\operatorname{Hom}_{\mathcal{S}_G}(U,G) \times_{\operatorname{Aut}_{\mathcal{S}_G}(U)} \operatorname{Hom}_{\mathcal{S}_G}(U,G) \to \mathcal{S}_{G,G} \,, \quad \phi \times_{\operatorname{Aut}_{\mathcal{S}_G}(U)} \psi \mapsto \Delta(\phi(U), \phi\psi^{-1}, \psi(U)) \,, \tag{23}$$

induce a bijection

$$\coprod_{U \in \tilde{\mathcal{S}}_G} \overline{\operatorname{Hom}}_{\mathcal{S}_G}(U,G) \times_{\operatorname{Aut}_{\mathcal{S}_G}(U)} \overline{\operatorname{Hom}}_{\mathcal{S}_G}(U,G) \to \mathcal{S}_{G,G}/(G \times G) \tag{24}$$

on the disjoint union, cf. (14). By the surjectivity of the map in (24) and by Proposition 1.14(c), we obtain $a = 0$. Thus, $\tilde{\sigma}_G^{\mathcal{S}}$ is injective.

    We still need to show that $\tilde{\sigma}_G^{\mathcal{S}}$ is multiplicative. But this is a straightforward variation of the proof of Part (a) in Theorem 5.5.

    (b) Note that the endomorphism ring in (20) is isomorphic to the set of integral matrices $(a_{[\phi],[\psi]})$ with rows and colums indexed by $\overline{\operatorname{Hom}}_{\mathcal{S}_G}(U,G)$ with the property that $a_{[\phi\alpha],[\psi\alpha]} = a_{[\phi],[\psi]}$ for all $\phi, \psi \in \operatorname{Hom}_{\mathcal{S}_G}(U,G)$ and all $\alpha \in \operatorname{Aut}_{\mathcal{S}_G}(U)$. Therefore, it has a standard basis indexed by the elements of the U-component in the coproduct in (24). Using the bijection in (24) and Proposition 1.14(c), it follows that, with respect to suitable bases, the map $\tilde{\sigma}_G^{\mathcal{S}}$ is represented by an upper triangular square matrix with diagonal entries equal to $[N_{G \times G}(L) : L]/|C_G(p_1(L))|$, where L runs through a transversal of $\mathcal{S}_{G,G}/(G \times G)$. This proves Part (b).   □

    Since endomorphism rings of semisimple artinian modules are semisimple, the following corollary is an immediate consequence of Theorem 5.7(b) and Maschke's Theorem. Independent proofs for the semisimplicity of $RB^{\Delta}(G,G)$, for a field R of characteristic 0, can for instance be found in [Bc96b, Corollaire 7] and [W, Theorem 9.6(1)].

5.8 Corollary *Let $G \in \mathcal{D}$ and let R be a field such that the numbers $|G|$ and $|\operatorname{Out}_{\mathcal{S}_G}(U)|$ are invertible in R for all $U \in \tilde{\mathcal{S}}_G$. Then the R-algebra $RB^{\mathcal{S}}(G,G)$ is semisimple. The isomorphism classes of simple $RB^{\mathcal{S}}(G,G)$-modules are in bijective correspondence with the pairs $(U,[V])$, where $U \in \tilde{\mathcal{S}}_G$ and $[V]$ is the isomorphism class of a simple $R\operatorname{Out}_{\mathcal{S}_G}(U)$-module V that occurs as a direct summand in the permutation module $R\overline{\operatorname{Hom}}_{\mathcal{S}_G}(U,G)$.*

5.9 Remark Assume again the notation from 5.6. We mention, without proof, that, for every $T \in \mathcal{T}$, there exists a ring isomorphism

$$\tilde{\tau}_{G,T}^{\mathcal{S}} \colon \operatorname{End}_{\mathbb{Z}\operatorname{Out}(T)}^{\mathcal{S}}(\mathbb{Z}\overline{\operatorname{Inj}}(T,G)) \to \bigoplus_{U \in \tilde{\mathcal{S}}_G(T)} \operatorname{End}_{\mathbb{Z}\operatorname{Out}_{\mathcal{S}_G}(U)}(\mathbb{Z}\overline{\operatorname{Hom}}_{\mathcal{S}_G}(U,G))$$

with the property that

$$(\tilde{\tau}_{G,T}^{\mathcal{S}})_{T \in \mathcal{T}} \circ \sigma_{G,G}^{\mathcal{S}} = \tilde{\sigma}_G^{\mathcal{S}} \, .$$

In the above direct sum, $\tilde{\mathcal{S}}_G(T)$ denotes a transversal of the $\mathcal{S}_G$-isomorphism classes of $\Sigma_G(T)$, the set of subgroups of G that are abstractly isomorphic to T. The homomorphism $\tilde{\tau}_{G,T}^{\mathcal{S}}$ is defined as follows: if $f \in \operatorname{End}_{\mathbb{Z}\operatorname{Out}(T)}^{\mathcal{S}}(\mathbb{Z}\overline{\operatorname{Inj}}(T,G))$ is represented by the matrix $(a_{[\lambda],[\mu]})$, then the U-component of $\tilde{\tau}_{G,T}^{\mathcal{S}}(f)$ is represented by the matrix $(b_{[\phi],[\psi]})$, where $b_{[\phi],[\psi]} := a_{[\phi\circ\theta],[\psi\circ\theta]}$ for some fixed isomorphism $\theta \colon T \xrightarrow{\sim} U$. This definition is independent of the choice of θ. We leave the verifications of the properties of $\tilde{\tau}_{G,T}^{\mathcal{S}}$ to the reader.

5.10 Remark Assume now that $\mathcal{D}$ is a set. Moreover, for simplicity, assume that for every $G \in \mathcal{D}$ and every subgroup $U \leqslant G$ there exists some $H \in \mathcal{D}$ with $H \cong U$, and assume that $\mathcal{S}_{G,H} = \Delta_{G,H}$ for all $G, H \in \mathcal{D}$. Assume further that R is a field such that $|G|$ and $|\operatorname{Out}(G)|$ are invertible in R, for every $G \in \mathcal{D}$. By 1.16, the module category $_{A_R^{\mathcal{D},s}}\mathsf{Mod}^*$ is then equivalent to the category of global Mackey functors on $\mathcal{D}$ over R. Note that, by Theorem 5.5, the collection of the maps $\sigma_{G,H}^{\mathcal{S}}$, $G, H \in \mathcal{D}$, defines an isomorphism of R-algebras

$$\sigma \colon A_R^{\mathcal{D},\mathcal{S}} \xrightarrow{\sim} \bigoplus_{T \in \mathcal{T}} \bigoplus_{G,H \in \mathcal{D}} \operatorname{Hom}_{R\operatorname{Out}(T)}(R\overline{\operatorname{Inj}}(T,H), R\overline{\operatorname{Inj}}(T,G)) \, ,$$

where the latter direct sum has componentwise multiplication with respect to T. Thus, the latter algebra is the direct sum of the R-algebras

$$\tilde{A}_{T,R}^{\mathcal{D},\mathcal{S}} := \bigoplus_{G,H \in \mathcal{D}} \operatorname{Hom}_{R\operatorname{Out}(T)}(R\overline{\operatorname{Inj}}(T,H), R\overline{\operatorname{Inj}}(T,G))$$

with multiplication of two components given by composition of homomorphisms if they are composable, and by the 0-product otherwise. From this point of view one could quickly show that the category $\mathsf{Func}_R^{\mathcal{D},\mathcal{S}}$ of global Mackey functors on $\mathcal{D}$ over R is semisimple and that the isomorphism classes of simple objects in $\mathsf{Func}_R^{\mathcal{D},\mathcal{S}}$ are parametrized by pairs $(T, [V])$, where $T \in \mathcal{T}$ such that $\mathcal{D}$ contains a group isomorphic to T, and were $[V]$ is the isomorphism class of a simple $R\operatorname{Out}(T)$-module V, cf. [W, Section 9].

6   The multiplicative structure of $\tilde{B}^{\triangleleft}(G,G)$

In this section we will study the structure of the ghost ring $\tilde{B}^{\triangleleft}(G,G)$. We show that it has a natural grading $\tilde{B}^{\triangleleft}(G,G) = \bigoplus_{n \geqslant 0} \tilde{B}_n^{\triangleleft}(G,G)$ and that the ideal $\bigoplus_{n \geqslant 1} \tilde{B}_n^{\triangleleft}(G,G)$ is nilpotent. This allows us to fully understand the simple modules of $R\tilde{B}^{\triangleleft}(G,G)$, provided that R is a field such that $|G|$ and $|\operatorname{Out}(U)|$, for all $U \leqslant G$, are invertible in R. These simple modules are then the same as those for $R\tilde{B}^{\Delta}(G,G)$.

Again, we will prove more general results, by considering the following situation: assume that $\mathcal{D}$ is a class of finite groups and that, for each choice of $G, H \in \mathcal{D}$, we are given a subset $\mathcal{S}_{G,H} \subseteq \triangleleft_{G,H}$ of

subgroups of $G \times H$ satisfying Condition (I) in Hypothesis 1.13. With these assumptions, $B^{\mathcal{S}}(G,H)$ is a subgroup of $B^{\triangleleft}(G,H)$, for $G, H \in \mathcal{D}$. Throughout this section, R denotes a commutative ring.

6.1 *A grading on $\tilde{B}^{\mathcal{S}}(G,H)$.* For a finite group G let $l(G)$ denote the composition length of G. For finite groups $G, H \in \mathcal{D}$ and $n \in \mathbb{N}_0$, we define

$$E^{\mathcal{S}}_{G,H,n} := \{(U, \alpha, V) \in E^{\mathcal{S}}_{G,H} \mid l(\ker(\alpha)) = n\}$$

and

$$A^{\mathcal{S}}_n(G,H) := \langle (U, \alpha, V) \mid (U, \alpha, V) \in E^{\mathcal{S}}_{G,H,n} \rangle_{\mathbb{Z}} \subseteq A^{\mathcal{S}}(G,H)\,.$$

Since $E^{\mathcal{S}}_{G,H}$ is the disjoint union of the subsets $E^{\mathcal{S}}_{G,H,n}$, we obtain a direct-sum decomposition

$$A^{\mathcal{S}}(G,H) = \bigoplus_{n \geqslant 0} A^{\mathcal{S}}_n(G,H)\,.$$

Clearly, $A^{\mathcal{S}}_n(G,H)$ is a $G \times H$-invariant subgroup of $A^{\mathcal{S}}(G,H)$, and we obtain direct-sum decompositions

$$\tilde{B}^{\mathcal{S}}(G,H) = \bigoplus_{n \geqslant 0} \tilde{B}^{\mathcal{S}}_n(G,H) \quad \text{and} \quad R\tilde{B}^{\mathcal{S}}(G,H) = \bigoplus_{n \geqslant 0} R\tilde{B}^{\mathcal{S}}_n(G,H)\,, \tag{25}$$

where $\tilde{B}^{\mathcal{S}}_n(G,H) := A^{\mathcal{S}}_n(G,H)^{G \times H} = \tilde{B}^{\mathcal{S}}(G,H) \cap A^{\mathcal{S}}_n(G,H)$. Note that

$$R\tilde{B}^{\mathcal{S}}_0(G,H) = R\tilde{B}^{\Delta(\mathcal{S})}(G,H)\,,$$

where $\Delta(\mathcal{S})_{G,H} := \mathcal{S}_{G,H} \cap \Delta_{G,H}$ for $G, H \in \mathcal{D}$.

Recall from 5.6 that we can define a category $\Delta(\mathcal{S})_G$ whose objects are the subgroups of G and whose morphism sets are determined by the groups in $\Delta(\mathcal{S})_{G,G}$. In the following, the Jacobson radical of any ring A will be denoted by $J(A)$.

6.2 Lemma *Let $G, H, K \in \mathcal{D}$.*
(a) For any $m, n \in \mathbb{N}_0$ one has

$$R\tilde{B}^{\mathcal{S}}_m(G,H) \cdot_H R\tilde{B}^{\mathcal{S}}_n(H,K) \subseteq R\tilde{B}^{\mathcal{S}}_{m+n}(G,K)\,.$$

(b) The decomposition in (25) provides the R-algebra $R\tilde{B}^{\mathcal{S}}(G,G)$ with the structure of a graded R-algebra such that $R\tilde{B}^{\mathcal{S}}_0(G,G) = R\tilde{B}^{\Delta(\mathcal{S})}(G,G)$.
(c) Assume that R is a field such that $|G|$ and $|\mathrm{Out}_{\Delta(\mathcal{S})_G}(U)|$ are units in R, for every subgroup $U \leqslant G$. Then

$$J(R\tilde{B}^{\mathcal{S}}(G,G)) = \bigoplus_{n \geqslant 1} R\tilde{B}^{\mathcal{S}}_n(G,G)\,.$$

In particular, one has a decomposition

$$R\tilde{B}^{\mathcal{S}}(G,G) = R\tilde{B}^{\Delta(\mathcal{S})}(G,G) \oplus J(R\tilde{B}^{\mathcal{S}}(G,G))\,.$$

Proof (a) It suffices to show that

$$A_m^{\lhd}(G,H) \cdot_H A_n^{\lhd}(H,K) \subseteq A_{m+n}^{\lhd}(G,H) \,.$$

So let $(U,\alpha,V) \in E_{G,H,m}$ and let $(V,\beta,W) \in E_{H,K,n}$. We need to show that $l(\ker(\alpha\beta)) = l(\ker(\alpha)) + l(\ker(\beta))$. But this follows from the short exact sequence

$$1 \longrightarrow \ker(\beta) \longrightarrow \ker(\alpha\beta) \overset{\beta}{\longrightarrow} \ker(\alpha) \longrightarrow 1$$

where the second arrow is the inclusion map.

(b) This follows immediately from Part (a).

(c) By Part (a), the subspace $I := \bigoplus_{n \geqslant 1} R\tilde{B}_n^{\mathcal{S}}(G,G)$ is an ideal of $R\tilde{B}^{\mathcal{S}}(G,G)$. Since $R\tilde{B}_n^{\mathcal{S}}(G,G) = 0$ for $n > \max\{l(H) \mid H \leqslant G\}$, Part (a) also implies that this is a nilpotent ideal, therefore contained in $J(R\tilde{B}^{\mathcal{S}}(G,G))$. On the other hand, the factor algebra modulo the ideal I is isomorphic to $R\tilde{B}_0^{\mathcal{S}}(G,G) = R\tilde{B}^{\Delta(\mathcal{S})}(G,G)$, which is semisimple by Corollary 5.8. This implies $J(R\tilde{B}^{\mathcal{S}}(G,G)) \subseteq I$ and we have equality. $\qquad\square$

6.3 *A grading on $RB^{\mathcal{S}}(G,G)$.* Let $G,H \in \mathcal{D}$ and let $n \in \mathbb{N}_0$. We define the subgroup

$$B_n^{\mathcal{S}}(G,H) \subseteq B^{\mathcal{S}}(G,H)$$

as the subset of those elements a in $B^{\mathcal{S}}(G,H)$ satisfying $\Phi_{\lhd(U,\alpha,V)}(a) = 0$ for all $(U,\alpha,V) \in \bigcup_{i \in \mathbb{N}_0 \smallsetminus \{n\}} E_{G,H,i}$.

Now assume that $|G \times H|$ is a unit in R. Then the isomorphism in Theorem 4.7(b) induces an isomorphism

$$\rho_{G,H}^{\mathcal{S}} \colon RB_n^{\mathcal{S}}(G,H) \overset{\sim}{\longrightarrow} R\tilde{B}_n^{\mathcal{S}}(G,H)$$

and we obtain a direct sum decomposition

$$RB^{\mathcal{S}}(G,H) = \bigoplus_{n \geqslant 0} RB_n^{\mathcal{S}}(G,H) \tag{26}$$

with

$$RB_0^{\mathcal{S}}(G,H) = RB^{\Delta(\mathcal{S})}(G,H) \,.$$

It seems worth mentioning that, in general, the sum $\sum_{n \geqslant 0} B_n^{\mathcal{S}}(G,H)$ is a proper subgroup of $B^{\mathcal{S}}(G,H)$. In fact, if G is a cyclic group of order 2 then $B_0^{\lhd}(G,G) + B_1^{\lhd}(G,G)$ is equal to the set of elements $a[G \times G/\Delta(G)] + b[G \times G/1] + c[G \times G/1 \times G]$ with $a,b \in \mathbb{Z}$ and $c \in 2\mathbb{Z}$. This is a subgroup of $B^{\lhd}(G,G)$ of index 2.

The following theorem is now immediate from Lemma 6.2 and the fact that the isomorphism $\rho_{G,H}^{\mathcal{S}}$ respects the tensor product construction of bisets, cf. Theorem 4.7(a).

6.4 Theorem *Let $G,H,K \in \mathcal{D}$.*

(a) *For any $m,n \in \mathbb{N}_0$ one has*

$$RB_m^{\mathcal{S}}(G,H) \cdot_H RB_n^{\mathcal{S}}(H,K) \subseteq RB_{m+n}^{\mathcal{S}}(G,K) \,.$$

(b) *Assume that $|G|$ is invertible in R. Then the R-algebra $RB^{\mathcal{S}}(G,G)$ is a graded R-algebra, with the grading given in (26).*

(c) *Assume that R is a field and that $|G|$ and $|\text{Out}_{\Delta(\mathcal{S})_G}(U)|$ are invertible in R, for every subgroup $U \leqslant G$. Moreover set $J := J(RB^{\mathcal{S}}(G,G))$. One has $J = \bigoplus_{n\geqslant 1} RB_n^{\mathcal{S}}(G,G)$, and J consists of precisely those elements $a \in RB^{\mathcal{S}}(G,G)$ satisfying $\Phi_{\Delta(U,\alpha,V)}(a) = 0$ for all $(U,\alpha,V) \in I_{G,H}^{\Delta(\mathcal{S})}$. In particular, one has*

$$RB^{\mathcal{S}}(G,G) = RB^{\Delta(\mathcal{S})}(G,G) \oplus J.$$

The following corollary is an immediate consequence of Theorem 6.4(c).

6.5 Corollary *Let $G \in \mathcal{D}$ and let R be a field such that $|G|$ and $|\text{Out}_{\Delta(\mathcal{S})_G}(U)|$ are units in R, for every subgroup $U \leqslant G$. Then the isomorphism classes of simple $RB^{\mathcal{S}}(G,G)$-modules and the isomorphism classes of simple $RB^{\Delta(\mathcal{S})}(G,G)$-modules are in natural bijective correspondence. More precisely, the correspondence is given by restriction from $RB^{\mathcal{S}}(G,G)$ to the subalgebra $RB^{\Delta(\mathcal{S})}(G,G)$. Its inverse is given by inflation from $RB^{\Delta(\mathcal{S})}(G,G)$ to $RB^{\mathcal{S}}(G,G)$ with respect to the ideal $\bigoplus_{n\geqslant 1} RB_n^{\mathcal{S}}(G,G)$.*

For the proof of the next theorem we first need a well-known result about Hecke algebras.

6.6 Lemma *Let R be a field of positive characteristic p, let G be a finite group and let H be a subgroup of G such that $|H|$ is not divisible by p but $|G|$ is divisible by p. Then $J(\text{End}_{RG}(\text{Ind}_H^G(R))) \neq \{0\}$.*

Proof Write e for the idempotent $|H|^{-1} \sum_{h\in H} h \in RG$. Note that $\text{Ind}_H^G(R) = R \otimes_{RH} RG$ is isomorphic to eRG as right RG-modules and that $\text{End}_{RG}(\text{Ind}_H^G(R))$ is isomorphic to $eRGe$ as R-algebras. Furthermore, set $a := \sum_{g\in G} g \in RG$. Then $0 \neq a = ea = eae \in eRGe$, and Ra is a non-zero ideal of $eRGe$. Since $a^2 = |G|a = 0$, we have $Ra \subseteq J(eRGe)$. □

The following theorem gives a criterion for the semisimplicity of $RB^{\mathcal{S}}(G,G)$ for an arbitrary field R. It can also be interpreted as a converse of the semisimplicity result in Corollary 5.8. For what follows, recall from 5.6 the definition of the category $\mathcal{S}_G$.

6.7 Theorem *Let R be a field, let $G \in \mathcal{D}$ be a non-trivial group and let $\tilde{\mathcal{S}}_G$ be a transversal of the $\mathcal{S}_G$-isomorphism classes of subgroups of G. Then the following statements are equivalent:*

(i) *The R-algebra $RB^{\mathcal{S}}(G,G)$ is semisimple.*

(ii) *One has $\mathcal{S}_{G,G} \subseteq \Delta_{G,G}$ and the numbers $|G|$ and $|\text{Out}_{\mathcal{S}_G}(U)|$, for all $U \in \tilde{\mathcal{S}}_G$, are invertible in R.*

Proof The statement in (ii) implies the statement in (i), by Corollary 5.8.

Next assume that $RB^{\mathcal{S}}(G,G)$ is semisimple. We first show that $|G|$ is invertible in R. Assume that this is not the case and consider the element $a := [G \times G] = [(G \times G)/\{1\}] \in RB^{\mathcal{S}}(G,G)$, the class of the regular $G \times G$-set. Let $b \in RB^{\mathcal{S}}(G,G)$ be arbitrary. Since the trivial subgroup $\{1\} \leqslant G \times G$ satisfies $L * \{1\} = \{1\}$ for every $L \in \triangleleft_{G,G}$, the multiplication formula in Proposition 1.10 implies that $b \cdot_G a = ra$ for some $r \in R$. Moreover the multiplication formula implies that $a \cdot_G a = |G|a = 0$. Thus, in the ring $RB^{\mathcal{S}}(G,G)$ we have $(1 + ra)(1 - ba) = (1 + ra)(1 - ra) = 1 - r^2 a^2 = 1$. This shows that $1 - ba$ has a left inverse. Since b was arbitrary, this shows that $a \in J(RB^{\mathcal{S}}(G,G))$. This is a contradiction to the semisimplicity of $RB^{\mathcal{S}}(G,G)$. Therefore, we have proved that $|G|$ is invertible in R.

Now Theorem 6.4(c) applies, and we derive that $\mathcal{S}_{G,G} \subseteq \Delta_{G,G}$.

With this established, Theorem 5.7(b) applies, and we obtain that $RB^{\mathcal{S}}(G,G)$ is a direct product of the R-algebras $\mathrm{End}_{R\mathrm{Out}_{\mathcal{S}_G}(U)}(R\overline{\mathrm{Hom}}_{\mathcal{S}_G}(U,G))$, where U varies over $\tilde{\mathcal{S}}_G$. Therefore, also these endomorphism algebras are semisimple. Now fix $U \in \tilde{\mathcal{S}}_G$ and let $\phi\colon U \to G$ be the inclusion map, which is contained in $\mathrm{Hom}_{\mathcal{S}_G}(U,G)$. Let $e \in \mathrm{End}_{R\mathrm{Out}_{\mathcal{S}_G}(U)}(R\overline{\mathrm{Hom}}_{\mathcal{S}_G}(U,G)) =: E$ denote the natural projection map onto the R-span of the $\mathrm{Out}_{\mathcal{S}_G}(U)$-orbit of $[\phi]$. Then e is an idempotent and eEe is also a semisimple R-algebra. But the latter R-algebra is isomorphic to the endomorphism ring of the transitive permutation $R\mathrm{Out}_{\mathcal{S}_G}(U)$-module whose basis is the orbit of $[\phi]$. Moreover, the stabilizer of $[\phi]$ in $\mathrm{Out}_{\mathcal{S}_G}(U)$ is isomorphic to $N_G(U)/(UC_G(U))$. So its order is a divisor of $|G|$, and hence also invertible in R. Thus, by Lemma 6.6, $|\mathrm{Out}_{\mathcal{S}_G}(U)|$ must be invertible in R, and the proof of the theorem is complete. $\qquad\square$

The following corollary extends a computational result of Webb (cf. [W, Theorem 9.6(2)]) for the group of order 2 and the case $\mathcal{S}_{G,G} = \lhd_{G,G}$.

6.8 Corollary *Let R be a non-zero commutative ring, let $G \in \mathcal{D}$, and assume that $\mathcal{S}_{G,G} \nsubseteq \Delta_{G,G}$. Then $RB^{\mathcal{S}}(G,G)$ is not semisimple.*

Proof Assume, for a contradiction, that $RB^{\mathcal{S}}(G,G)$ is semisimple. Let $\bar{R}$ be the factor ring of R modulo some maximal ideal. Then $\bar{R}B^{\mathcal{S}}(G,G)$ is a factor ring of $RB^{\mathcal{S}}(G,G)$. Since $RB^{\mathcal{S}}(G,G)$ is assumed to be semisimple, so is $\bar{R}B^{\mathcal{S}}(G,G)$. Theorem 6.7 now implies $\mathcal{S}_{G,G} \subseteq \Delta_{G,G}$, a contradiction. $\qquad\square$

6.9 Remark Let again $\mathcal{D}$ be a set. Assume for simplicity that R is a field of characteristic 0, that $\mathcal{S}_{G,H} = \lhd_{G,H}$ for all $G, H \in \mathcal{D}$, and that $\mathcal{D}$ is closed under taking subgroups in the sense that every subgroup of a group $G \in \mathcal{D}$ is isomorphic to a group in $\mathcal{D}$. Then $\Delta(\mathcal{S})_{G,H} = \Delta_{G,H}$ for all $G, H \in \mathcal{D}$. Theorem 6.4 implies that the R-algebra $\tilde{A}_R^{\mathcal{D},\mathcal{S}} = \tilde{A}_R^{\mathcal{D},\lhd}$ is graded by

$$\tilde{A}_R^{\mathcal{D},\lhd} = \bigoplus_{n \geqslant 0} \tilde{A}_{n,R}^{\mathcal{D},\lhd}$$

with

$$\tilde{A}_{n,R}^{\mathcal{D},\lhd} := \bigoplus_{G,H \in \mathcal{D}} R\tilde{B}_n^{\lhd}(G,H) \,,$$

for $n \geqslant 0$. From Lemma 6.2(c) it is also straightforward to see that every simple object in $_{\tilde{A}_R^{\mathcal{D},\lhd}}\mathsf{Mod}^*$ is annihilated by the ideal $\tilde{J}_R^{\mathcal{D},\lhd} := \bigoplus_{n \geqslant 1} \tilde{A}_{n,R}^{\mathcal{D},\lhd}$. Note that the subalgebra $\tilde{A}_{0,R}^{\mathcal{D},\lhd} = \tilde{A}_R^{\mathcal{D},\Delta}$ is the algebra considered in Remark 5.10. Now the decomposition

$$\tilde{A}_R^{\mathcal{D},\lhd} = \tilde{A}_R^{\mathcal{D},\Delta} \oplus \tilde{J}_R^{\mathcal{D},\lhd},$$

together with the category equivalences in 1.16 and Remark 4.9, implies that the isomorphism classes of simple objects in $\mathsf{Func}_R^{\mathcal{D},\lhd}$ and those in $\mathsf{Func}_R^{\mathcal{D},\Delta}$ are in natural bijective correspondence via restriction and inflation with respect to the above direct sum decomposition.

Moreover, one has a filtration of $\tilde{A}_R^{\mathcal{S},\lhd}$ by the ideals $\tilde{J}_n^{\mathcal{S},\lhd} := \bigoplus_{i \geqslant n} \tilde{A}_{i,R}^{\mathcal{S},\lhd}$, for $n \geqslant 0$. This leads to a natural filtration of every functor in $\mathsf{Func}_R^{\mathcal{D},\lhd}$ with successive semisimple factors.

7 Fusion systems

Throughout this section, p denotes a prime number, $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ denotes the localization of $\mathbb{Z}$ with respect to the prime ideal $(p) = p\mathbb{Z}$, and S a finite p-group. As before, R is a commutative ring.

In this section we will show that fusion systems $\mathcal{F}$ on S are in bijective correspondence with subsets $\mathcal{S}_{S,S}$ of $\Delta_{S,S}$ that satisfy the Axioms (i)–(v) in Hypothesis 1.13. Thus, we can consider the ring $B^{\mathcal{F}}(S, S) = B^{\mathcal{S}}(S, S)$ as an invariant of the fusion system $\mathcal{F}$. We identify the characteristic idempotent $\omega_{\mathcal{F}}$ of a saturated fusion system $\mathcal{F}$ in the ghost ring $\mathbb{Z}_{(p)} \tilde{B}^{\mathcal{F}}(S, S)$ and are able to compute its marks, i.e., its numbers of fixed points with respect to the subgroups in $\Delta_{S,S}$. Moreover, we extend the bijection between the set of saturated fusion systems on S and a certain set of idempotents in $\mathbb{Z}_{(p)} B^{\Delta}(S, S)$, which was observed by Ragnarsson and Stancu in [RS], to a bijection between the set of all fusion systems on S and a certain set of idempotents in $\mathbb{Q}B^{\Delta}(S, S)$.

Recall that whenever P and Q are subgroups of S, we denote by $\mathrm{Hom}_S(P, Q)$ the set of homomorphisms $P \to Q$ that are induced by conjugations with elements in S. Moreover, we again denote by $\mathrm{Inj}(P, Q)$ the set of all injective group homomorphisms $P \to Q$.

First we recall the definition of a fusion system and of a saturated fusion system on S, cf. [L] for instance.

7.1 Definition (a) A *fusion system on S* is a category $\mathcal{F}$ whose objects are the subgroups of S, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ satisfy the following conditions:

(i) If $P, Q \leqslant S$ then $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$.

(ii) If $P, Q \leqslant S$ and if $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ then the resulting isomorphism $P \xrightarrow{\sim} \phi(P)$ as well as its inverse are morphisms in $\mathcal{F}$.

The composition of morphisms in $\mathcal{F}$ is the usual composition of maps. Whenever subgroups P and Q of S are isomorphic in $\mathcal{F}$, we write $P =_{\mathcal{F}} Q$.

(b) For a fusion system $\mathcal{F}$ on S one introduces the following three notions:

(i) A subgroup P of S is called *fully $\mathcal{F}$-centralized* if, for every $Q \leqslant S$ with $Q =_{\mathcal{F}} P$, one has $|C_S(Q)| \leqslant |C_S(P)|$.

(ii) A subgroup P of S is called *fully $\mathcal{F}$-normalized* if, for every $Q \leqslant S$ with $Q =_{\mathcal{F}} P$, one has $|N_S(Q)| \leqslant |N_S(P)|$.

(iii) For every $P \leqslant S$ and every $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, one sets

$$N_{\phi} := \{ y \in N_S(P) \mid \exists z \in N_S(\phi(P)) : \phi(^y u) = {}^z \phi(u) \; \forall u \in P \}.$$

(c) A fusion system $\mathcal{F}$ on S is called *saturated* if the following axioms are satisfied:

(i) (*Sylow Axiom*) $\mathrm{Aut}_S(S) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(S))$.

(ii) (*Extension Axiom*) Every morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ such that $\phi(P)$ is fully $\mathcal{F}$-normalized extends to a morphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_{\phi}, S)$.

7.2 *Fusion systems on S and subsystems of $\Delta_{S,S}$.* (a) Suppose that $\mathcal{F}$ is a fusion system on S, and consider
$$\mathcal{S} := \mathcal{S}(\mathcal{F}) := \{ \Delta(\phi(P), \phi, P) \mid P \leqslant S, \phi \in \mathrm{Hom}_{\mathcal{F}}(P, S) \}.$$

Then the class $\mathcal{D} := \{S\}$, together with the set $\mathcal{S}$, satisfies Conditions (I) and (II) in Hypothesis 1.13. This follows immediately from the definition of a fusion system. Thus, we can introduce the notation $B^{\mathcal{F}}(S, S)$ for $B^{\mathcal{S}}(S, S)$.

(b) Suppose, conversely, we are given a set $\mathcal{S} \subseteq \Delta_{S,S}$ such that $\mathcal{D} := \{S\}$, together with this set $\mathcal{S}$, satisfies Conditions (I) and (II) in Hypothesis 1.13. Then we define a category $\mathcal{F} = \mathcal{F}(\mathcal{S})$ as follows: the objects in $\mathcal{F}$ are the subgroups of S. For $P, Q \leqslant S$, we define

$$\mathrm{Hom}_{\mathcal{F}}(P, Q) := \{\iota \circ \phi \mid \Delta(\phi(P), \phi, P) \in \mathcal{S} \text{ and } \phi(P) \leqslant Q\},$$

where $\iota : \phi(P) \to Q$ denotes the inclusion map. The composition of morphisms in $\mathcal{F}$ shall be the usual composition of maps. Then $\mathcal{F}$ is a fusion system on S.

We denote the set of fusion systems on S by $\mathrm{Fus}(S)$. This is a finite poset, with the partial order given by the subcategory relation. The set of subsets $\mathcal{S}$ of $\Delta_{S,S}$ such that the class $\mathcal{D} = \{S\}$ together with $\mathcal{S}_{S,S} = \mathcal{S}$ satisfies Conditions (I) and (II) in Hypothesis 1.13 will be denoted by $\mathrm{Sys}(S)$. This is a finite poset with respect to inclusion. The following theorem is an easy exercise, and is left to the reader.

7.3 Theorem *The constructions in 7.2 are mutually inverse isomorphisms between the partially ordered sets* $\mathrm{Fus}(S)$ *and* $\mathrm{Sys}(S)$.

7.4 Examples (a) The following is a standard example of a saturated fusion system. Let G be a group such that $S \in \mathrm{Syl}_p(G)$. Then the category $\mathcal{F}_S(G)$ with objects given by the set of subgroups of S and with morphisms

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P, Q) := \mathrm{Hom}_G(P, Q), \quad (P, Q \leqslant S)$$

is a saturated fusion system on S, called the *fusion system of G on S*. For a proof, see [L]. The corresponding set $\mathcal{S}$ is thus given as

$$\mathcal{S} = \{\Delta(\phi(P), \phi, P) \mid P \leqslant S, \ \phi \in \mathrm{Hom}_G(P, S)\},$$

or as the set of subgroups $L \in \Delta_{S,S}$ such that $L \leqslant_{G \times G} \Delta(S)$.

(b) Consider the alternating group $\mathfrak{A}_4$ of degree 4, and let S be the unique Sylow 2-subgroup of $\mathfrak{A}_4$. That is, $S = \langle (1,2)(3,4), (1,4)(2,3) \rangle$ is a Klein four-group. The automorphism group $\mathrm{Aut}(S)$ of S is isomorphic to the symmetric group $\mathfrak{S}_3$ of degree 3. Let $A \trianglelefteq \mathrm{Aut}(S)$ with $A \cong \mathfrak{A}_3$. Then the saturated fusion system $\mathcal{F} := \mathcal{F}_S(\mathfrak{A}_4)$ corresponds to the set

$$\{\Delta(\phi(P), \phi|_P, P) \mid P \leqslant S, \ \phi \in A\} \subseteq \Delta_{S,S}.$$

Namely,

$$\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathfrak{A}_4}(S) \cong N_{\mathfrak{A}_4}(S)/C_{\mathfrak{A}_4}(S) = \mathfrak{A}_4/S \cong \mathfrak{A}_3,$$

so that $\mathrm{Aut}_{\mathcal{F}}(S) = A$. Now suppose that $P \leqslant S$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. Then, since S is abelian, we have $N_\psi = S$. Hence the Extension Axiom forces $\mathrm{Hom}_{\mathcal{F}}(P, S) = \{\psi|_P \mid \psi \in A\}$ so that $\{\Delta(\psi(P), \psi|_P, P) \mid P \leqslant S, \ \psi \in A\} = \mathcal{S}(\mathcal{F})$.

(c) Let S and A be as in Part (b) above, and this time consider the set

$$\mathcal{S} := \{\Delta(S)\} \cup \{\Delta(\phi(P), \phi|_P, P) \mid P < S, \ \phi \in A\} \subseteq \Delta_{S,S}.$$

The class $\mathcal{D} := \{S\}$, together with the set $\mathcal{S}$, obviously satisfies Conditions (I) and (II) in Hypothesis 1.13. Thus $\mathcal{S}$ gives rise to a fusion system $\mathcal{F} := \mathcal{F}(\mathcal{S})$ on S. However, $\mathcal{F}$ is not saturated. To see this, let $Q := \langle (1,2)(3,4) \rangle$ and $P := \langle (1,3)(2,4) \rangle$. Let further $\phi \in A$ be the automorphism of S induced by conjugation with $(1,2,3) \in \mathfrak{A}_3$. By definition, the restriction $\psi := \phi|_P$ belongs to $\mathrm{Hom}_{\mathcal{F}}(P, Q)$. Since S is abelian, we have $N_\psi = S$. Since $\mathrm{Aut}_{\mathcal{F}}(S) = \{\mathrm{id}_S\}$, the morphism ψ does not extend to N_ψ, and $\mathcal{F}$ is, therefore, not saturated.

In [RS], K. Ragnarsson and R. Stancu established a bijective correspondence between saturated fusion systems on S and certain idempotents in the $\mathbb{Z}_{(p)}$-algebra $\mathbb{Z}_{(p)} B^\Delta(S, S)$. We next aim to use our results from Section 5 to extend the bijection of Ragnarsson–Stancu to a bijection between the set of all fusion systems on S and certain idempotents in the $\mathbb{Q}$-algebra $\mathbb{Q} B^\Delta(S, S)$. In order to do so, we recall the basic notions from [RS] needed in subsequent statements.

7.5 Consider the natural $\mathbb{Z}$-bilinear map $- \times -$, given by

$$- \times -\colon B^\Delta(S, S) \times B^\Delta(S, S) \to B^\Delta(S \times S, S \times S), ([X], [Y]) \mapsto [X \times Y],$$

where X and Y are bifree (S, S)-bisets. We view $X \times Y$ as $(S \times S, S \times S)$-biset by $(s_1, s_2)(x, y)(s_1', s_2') = (s_1 x s_1', s_2 y s_2')$, for $s_1, s_2, s_1', s_2' \in S$ and $(x, y) \in X \times Y$. This map induces, for every commutative ring R, an R-bilinear map

$$RB^\Delta(S, S) \times RB^\Delta(S, S) \to RB^\Delta(S \times S, S \times S), ([X], [Y]) \mapsto [X \times Y].$$

Moreover, if X and Y are bifree (S, S)-bisets then $X \times Y$ carries, via restriction along the diagonal map $s \mapsto (s, s)$, a bifree $(S \times S, S)$-biset structure (respectively, a bifree $(S, S \times S)$-biset structure) with

$$(g, h)(x, y)k := (gxk, hyk) \quad (\text{respectively, } g(x, y)(h, k) := (gxh, gyk)),$$

for all $x \in X$, $y \in Y$, $g, h, k \in S$.

7.6 Definition Let $a \in RB^\Delta(S, S)$. We say that
(a) a is a *right Frobenius element* if the following equality holds in $RB^\Delta(S \times S, S)$:

$$a \times a = (a \times [S]) \cdot_S a. \tag{27}$$

(b) a is a *left Frobenius element* if the following equality holds in $RB^\Delta(S, S \times S)$:

$$a \times a = a \cdot_S (a \times [S]). \tag{28}$$

(c) a is a *Frobenius element* if it is a left Frobenius element and a right Frobenius element.

7.7 Remark Let $a \in RB^\Delta(S, S)$. Then the equality $a \times a = (a \times [S]) \cdot_S a$ is equivalent to the equality $a \times a = ([S] \times a) \cdot_S a$ in $RB^\Delta(S \times S, S)$. Similarly, the equality $a \times a = a \cdot_S (a \times [S])$ is equivalent to the equality $a \times a = a \cdot_S ([S] \times a)$ in $RB^\Delta(S, S \times S)$. This follows quickly by applying the natural function $X \times Y \xrightarrow{\sim} Y \times X$, $(x, y) \mapsto (y, x)$ for (S, S)-bisets X and Y, and the group isomorphism $S \times S \xrightarrow{\sim} S \times S$, $(s, t) \mapsto (t, s)$.

As an immediate consequence of Definition 7.6, we obtain

7.8 Proposition *Let R be a commutative ring, and let $a \in RB^\Delta(S, S)$. Then a is a right Frobenius element if and only if a° is a left Frobenius element.*

Our next aim is to establish a bijection between the set $\mathrm{Fus}(S)$ of fusion systems on S and a set $\mathrm{Idem}(S)$ of certain idempotents $\omega \in \mathbb{Q} B^\Delta(S, S)$. Recall from Subsection 1.11 the map $\Phi_L \colon B(S, S) \to \mathbb{Z}$, $[X] \mapsto |X^L|$, where $L \leqslant S \times S$ can be any subgroup and X can be any (S, S)-biset.

7.9 Definition We denote by $\mathrm{Idem}(S)$ the set of idempotents $\omega \in \mathbb{Q}B^\Delta(S,S)$ satisfying the following properties:

(i) ω is a Frobenius element,

(ii) $\mathrm{Fix}(\omega) := \{L \in \Delta_{S,S} \mid \Phi_L(\omega) \neq 0\}$ is closed under taking subgroups,

(iii) $\Delta(S) \in \mathrm{Fix}(\omega)$.

We will construct the bijection between $\mathrm{Idem}(S)$ and $\mathrm{Fus}(S)$ in several steps, and we begin by recalling from [RS] a criterion for $a \in \mathbb{Q}B^\Delta(S,S)$ being a Frobenius element. The result [RS, Lemma 7.4] is formulated for $\mathbb{Z}_{(p)}$, and one cannot directly lift the statement to $\mathbb{Q}$, since the two sides of the equation defining the Frobenius property are not linear in the element a. However, the proof of Lemma 7.4 in [RS] implies that the result remains true when replacing $\mathbb{Z}_{(p)}$ by $\mathbb{Q}$, or any commutative ring R. We provide these arguments in the proof of the following proposition for the reader's convenience.

7.10 Proposition (cf. [RS], §7) *Let $a \in \mathbb{Q}B^\Delta(S,S)$. Then the following hold:*

(a) The element a is a right Frobenius element if and only if, for every $P \leqslant S$ and all $\phi, \psi \in \mathrm{Inj}(P,S)$, one has

$$\Phi_{\Delta(\phi(P),\phi,P)}(a)\Phi_{\Delta(\psi(P),\psi,P)}(a) = \Phi_{\Delta(\phi(P),\phi\psi^{-1},\psi(P))}(a)\Phi_{\Delta(\psi(P),\psi,P)}(a)\,.$$

(b) The element a is a left Frobenius element if and only if, for every $P \leqslant S$ and all $\phi, \psi \in \mathrm{Inj}(P,S)$, one has

$$\Phi_{\Delta(P,\phi^{-1},\phi(P))}(a)\Phi_{\Delta(P,\psi^{-1},\psi(P))}(a) = \Phi_{\Delta(\psi(P),\psi\phi^{-1},\phi(P))}(a)\Phi_{\Delta(P,\psi^{-1},\psi(P))}(a)\,.$$

Proof Suppose that $L \in \Delta_{S\times S, S}$. Then there exist $P \leqslant S$ and $\phi, \psi \in \mathrm{Inj}(P,S)$ such that $L = \Delta((\phi \times \psi)(P), \phi \times \psi, P)$, where $(\phi \times \psi)(P) := \{(\phi(g), \psi(g)) \mid g \in P\} \leqslant S \times S$.

The proof of [RS, Lemma 7.4] shows that, for all $a, b \in \mathbb{Q}B^\Delta(S,S)$ and all $L = \Delta((\phi \times \psi)(P), \phi \times \psi, P) \in \Delta_{S\times S, S}$, we have

$$\Phi_L(a \times b) = \Phi_{\Delta(\phi(P),\phi,P)}(a)\Phi_{\Delta(\psi(P),\psi,P)}(b), \tag{29}$$

and

$$\Phi_L((a \times [S]) \cdot_S b) = \Phi_{\Delta(\phi(P),\phi\psi^{-1},\psi(P))}(a)\Phi_{\Delta(\psi(P),\psi,P)}(b). \tag{30}$$

Specializing $a = b$ and using Proposition 1.14, Assertion (a) follows.

As for (b), let again $a \in \mathbb{Q}B^\Delta(S,S)$. By Proposition 7.8, a is a left Frobenius element if and only if a° is a right Frobenius element. By Part (a), this in turn is equivalent to requiring

$$\begin{aligned}
\Phi_{\Delta(P,\phi^{-1},\phi(P))}(a)\Phi_{\Delta(P,\psi^{-1},\psi(P))}(a) &= \Phi_{\Delta(\phi(P),\phi,P)}(a^\circ)\Phi_{\Delta(\psi(P),\psi,P)}(a^\circ) \\
&= \Phi_{\Delta(\phi(P),\phi\psi^{-1},\psi(P))}(a^\circ)\Phi_{\Delta(\psi(P),\psi,P)}(a^\circ) \\
&= \Phi_{\Delta(\psi(P),\psi\phi^{-1},\phi(P))}(a)\Phi_{\Delta(P,\psi^{-1},\psi(P))}(a),
\end{aligned}$$

for all $P \leqslant S$ and all $\phi, \psi \in \mathrm{Inj}(P,S)$. This settles (b). $\qquad\square$

7.11 *The idempotent $\omega_{\mathcal{F}}$.* Let $\mathcal{F}$ be a fusion system on S, and let $\tilde{\mathcal{F}}$ be a transversal for the $\mathcal{F}$-isomorphism classes of subgroups of S.

(a) Recall from Theorem 5.7 that we have a $\mathbb{Q}$-algebra isomorphism

$$\tilde{\sigma}^{\mathcal{F}} \colon \mathbb{Q}B^{\mathcal{F}}(S,S) \to \prod_{P \in \tilde{\mathcal{F}}} \mathrm{End}_{\mathbb{Q}\mathrm{Out}_{\mathcal{F}}(P)}(\mathbb{Q}\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)),$$

$$a \mapsto \left([\psi] \mapsto \sum_{[\phi] \in \overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)} \frac{\Phi_{\Delta(\phi(P),\phi\psi^{-1},\psi(P))}(a)}{|C_S(\phi(P))|}[\phi] \right)_{P}. \tag{31}$$

For every $P \in \tilde{\mathcal{F}}$ we define $\varepsilon_P \in \mathrm{End}_{\mathbb{Q}\mathrm{Out}_{\mathcal{F}}(P)}(\mathbb{Q}\overline{\mathrm{Hom}}_{\mathcal{F}}(P,S))$ such that

$$\varepsilon_P([\psi]) := \sum_{[\phi] \in \overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)} \frac{|S|}{|C_S(\phi(P))||\mathrm{Hom}_{\mathcal{F}}(P,S)|}[\phi], \tag{32}$$

for every $[\psi] \in \overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)$. We then define $\omega_{\mathcal{F}} \in \mathbb{Q}B^{\mathcal{F}}(S,S)$ via

$$\tilde{\sigma}^{\mathcal{F}}(\omega_{\mathcal{F}}) := (\varepsilon_P)_{P \in \tilde{\mathcal{F}}}. \tag{33}$$

(b) Let $\mathcal{S} := \mathcal{S}(\mathcal{F})$ and note (cf. (24)) that we have a bijection

$$\coprod_{P \in \tilde{\mathcal{F}}} \mathrm{Hom}_{\mathcal{F}}(P,S) \times_{\mathrm{Aut}_{\mathcal{F}}(P)} \mathrm{Hom}_{\mathcal{F}}(P,S) \to \mathcal{S},$$

$$\phi \times_{\mathrm{Aut}_{\mathcal{F}}(P)} \psi \mapsto \Delta(\phi(P), \phi\psi^{-1}, \psi(P)), \tag{34}$$

of $S \times S$-sets. Thus, by (31) and (32), the element $\omega_{\mathcal{F}} \in \mathbb{Q}B^{\Delta}(S,S)$ is characterized by

$$\Phi_L(\omega_{\mathcal{F}}) = \begin{cases} \frac{|S|}{|\mathrm{Hom}_{\mathcal{F}}(p_1(L),S)|} & \text{if } L \in \mathcal{S}, \\ 0 & \text{if } L \notin \mathcal{S}. \end{cases} \tag{35}$$

In particular, $\omega_{\mathcal{F}}^{\circ} = \omega_{\mathcal{F}}$ is symmetric, since $|\mathrm{Hom}_{\mathcal{F}}(P,S)| = |\mathrm{Hom}_{\mathcal{F}}(Q,S)|$ if $P =_{\mathcal{F}} Q$.

The following lemma gives a list of properties of the element $\omega_{\mathcal{F}}$, including a uniqueness statement.

7.12 Lemma *Let $\mathcal{F}$ be a fusion system on S, let $\mathcal{S} := \mathcal{S}(\mathcal{F})$, and let $\omega_{\mathcal{F}} \in \mathbb{Q}B^{\mathcal{F}}(S,S)$ be the element defined in (33). Then*
    (a) $\omega_{\mathcal{F}}$ is an idempotent in $\mathbb{Q}B^{\Delta}(S,S)$;
    (b) $\omega_{\mathcal{F}}$ is a Frobenius element;
    (c) $\mathrm{Fix}(\omega_{\mathcal{F}}) = \mathcal{S}$.
In particular, $\omega_{\mathcal{F}} \in \mathrm{Idem}(S)$.
    Moreover, $\omega_{\mathcal{F}}$ is the unique element in $\mathbb{Q}B^{\Delta}(S,S)$ satisfying Properties (a)–(c).

Proof (a) A quick computation shows that, for every $P \in \tilde{\mathcal{F}}$, the endomorphism ε_P is an idempotent. In fact this comes down to the equation

$$\sum_{[\phi] \in \overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)} \frac{|S|}{|C_S(\phi(P)||\mathrm{Hom}_{\mathcal{F}}(P,S)|} = 1,$$

37

which holds, since $C_S(\phi(P))$ is the stabilizer of ϕ in the S-set $\mathrm{Hom}_{\mathcal{F}}(P,S)$.

(b) This follows immediately from Equation (35) and Proposition 7.10.

(c) This follows immediately from Equation (35).

To show the uniqueness statement, let $\omega \in \mathbb{Q}B^{\Delta}(S,S)$ satisfy (a)–(c). Since ω satisfies (c), we obtain $\Phi_L(\omega) = 0$ for $L \notin \mathcal{S}$. We fix $P \in \tilde{\mathcal{F}}$. By (35), it suffices to show that

$$a_{\phi,\psi} := \Phi_{\Delta(\phi(P),\phi\psi^{-1},\psi(P))}(\omega) = \frac{|S|}{|\mathrm{Hom}_{\mathcal{F}}(P,S)|}, \tag{36}$$

for all $P \in \tilde{\mathcal{F}}$ and $\phi, \psi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. Here, we made use of the bijection in (34) and the equality $|\mathrm{Hom}_{\mathcal{F}}(\phi(P),S)| = |\mathrm{Hom}_{\mathcal{F}}(P,S)|$. We first show that $a_{\phi,\psi} = a_{\mathrm{id}_P,\mathrm{id}_P}$ for all $\phi, \psi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. Since $\Phi_{\Delta(\psi(P),\psi,P)}(\omega) \neq 0$ by Property (c), Proposition 7.10(a) implies that $a_{\phi,\psi} = a_{\phi,\mathrm{id}_P}$, since ω is right Frobenius. Similarly, replacing ϕ and ψ in the statement of Proposition 7.10(b) with id_P and ϕ, respectively, implies $a_{\mathrm{id}_P,\mathrm{id}_P} = a_{\phi,\mathrm{id}_P}$. Now we can abbreviate $a_{\phi,\psi}$ by a constant rational number c. Writing ζ_P for the P-component of $\tilde{\sigma}^{\mathcal{F}}(\omega)$, the endomorphism ζ_P satisfies

$$\zeta_P([\psi]) = \sum_{[\phi] \in \overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)} \frac{c}{|C_S(\phi(P))|}\,[\phi],$$

for all $\psi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. Since ω is an idempotent, so is ζ_P. Evaluating the equation $\zeta_P \circ \zeta_P = \zeta_P$ yields

$$\sum_{[\phi] \in \overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)} \frac{c^2}{|C_S(\phi(P))|} = c.$$

Since $c \neq 0$ by Property (c), and since $C_S(\phi(P))$ is the stabilizer of the element ϕ in the S-set $\mathrm{Hom}_{\mathcal{F}}(P,S)$, we obtain

$$c = \Big(\sum_{[\phi] \in \overline{\mathrm{Hom}}_{\mathcal{F}}(P,S)} \frac{1}{|C_S(\phi(P))|} \Big)^{-1} = \Big(\sum_{\phi \in \mathrm{Hom}_{\mathcal{F}}(P,S)} \frac{1}{|S|} \Big)^{-1} = \frac{|S|}{|\mathrm{Hom}_{\mathcal{F}}(P,S)|},$$

as desired in Equation (36). This completes the proof of the lemma. $\qquad\square$

7.13 Lemma *Let $\omega \in \mathbb{Q}B^{\Delta}(S,S)$ be a right Frobenius element such that $\Delta(S) \in \mathrm{Fix}(\omega)$ and such that $\mathrm{Fix}(\omega)$ is closed under taking subgroups. Then $\mathcal{S} := \mathrm{Fix}(\omega) \in \mathrm{Sys}(S)$.*

Proof We verify that $\mathcal{S}$ satisfies (i)–(v) in Hypothesis 1.13. By definition, $\mathcal{S}$ is closed under $S \times S$-conjugation, and by our hypotheses it is closed under taking subgroups and contains $\Delta(S)$. In particular, $\Phi_{\Delta(P)}(\omega) \neq 0$, for all $P \leqslant S$.

Let $\Delta(\psi(P),\psi,P) \in \mathcal{S}$, so that $\Phi_{\Delta(\psi(P),\psi,P)}(\omega) \neq 0$. Since ω is a right Frobenius element, Proposition 7.10(a) with $\varphi = \mathrm{id}_P$ gives

$$0 \neq \Phi_{\Delta(P)}(\omega) = \Phi_{\Delta(P,\psi^{-1},\psi(P))}(\omega),$$

thus $\Delta(\psi(P),\psi,P)^{\circ} = \Delta(P,\psi^{-1},\psi(P)) \in \mathcal{S}$.

It remains to show that if $\Delta(\phi(P),\phi,P) \in \mathcal{S}$ and $\Delta(\psi(Q),\psi,Q) \in \mathcal{S}$ then also $\Delta(\phi(P),\phi,P) * \Delta(\psi(Q),\psi,Q) \in \mathcal{S}_{S,S}$.

Suppose first that $\psi(Q) = P$. Since ω is a right Frobenius element, Proposition 7.10(a) implies $\Phi_{\Delta(\phi(P),\phi,P)}(\omega)\Phi_{\Delta(Q,\psi^{-1},\psi(Q))}(\omega) = \Phi_{\Delta(\phi(P),\phi\psi,Q)}(\omega)\Phi_{\Delta(Q,\psi^{-1},\psi(Q))}(\omega)$. As we have just shown, $\Phi_{\Delta(Q,\psi^{-1},\psi(Q))}(\omega) \neq 0$, so that $0 \neq \Phi_{\Delta(\phi(P),\phi,P)}(\omega) = \Phi_{\Delta(\phi(P),\phi\psi,Q)}(\omega)$, thus $\Delta(\phi(P),\phi,P) * \Delta(\psi(Q),\psi,Q) = \Delta(\phi(P),\phi\psi,Q) \in \mathcal{S}$.

In the general case we have

$$\Delta(\phi(P),\phi,P) * \Delta(\psi(Q),\psi,Q) = \Delta(\phi(P \cap \psi(Q)),\phi\psi,\psi^{-1}(P \cap \psi(Q)))$$
$$= \Delta(\phi(P \cap \psi(Q)),\phi,P \cap \psi(Q)) * \Delta(P \cap \psi(Q),\psi,\psi^{-1}(P \cap \psi(Q))).$$

Since $\mathcal{S}$ is closed under taking subgroups, our considerations in the special case above show that also $\Delta(\phi(P),\phi,P) * \Delta(\psi(Q),\psi,Q) \in \mathcal{S}$, and the proof is complete. $\quad\square$

7.14 Remark By Lemma 7.12 and Lemma 7.13 we obtain maps

$$f\colon \mathrm{Fus}(S) \to \mathrm{Idem}(S)\,, \quad \mathcal{F} \mapsto \omega_{\mathcal{F}}\,, \quad \text{and} \quad g\colon \mathrm{Idem}(S) \to \mathrm{Sys}(S)\,, \quad \omega \mapsto \mathrm{Fix}(\omega)\,.$$

Moreover, we write

$$h\colon \mathrm{Fus}(S) \xrightarrow{\ \sim\ } \mathrm{Sys}(S)\,, \quad \mathcal{F} \mapsto \mathcal{S}(\mathcal{F})\,,$$

for the bijection in Theorem 7.3. Then we obtain a triangle diagram

$$
\begin{array}{ccc}
\mathrm{Fus}(S) & \xrightarrow{\ f\ } & \mathrm{Idem}(S) \\
& h \searrow \quad \swarrow g & \\
& \mathrm{Sys}(S) &
\end{array}
$$

Summarizing our previous considerations, we obtain the following theorem.

7.15 Theorem *The maps f, g and h in Remark 7.14 are bijections, and the triangle diagram in Remark 7.14 is commutative.*

Proof We already know from Theorem 7.3 that h is a bijection.

Let $\mathcal{F}$ be a fusion system on S. By Lemma 7.12(c), we have $g(f(\mathcal{F})) = g(\omega_{\mathcal{F}}) = \mathrm{Fix}(\omega_{\mathcal{F}}) = \mathcal{S}(\mathcal{F}) = h(\mathcal{F})$. Therefore, $g \circ f = h$. Since h is bijective, f is injective and g is surjective.

It suffices now to show that f is surjective. So let $\omega \in \mathrm{Idem}(S)$ and set $\mathcal{F} := h^{-1}(g(\omega))$. Then $\mathrm{Fix}(\omega) = g(\omega) = h(\mathcal{F}) = \mathcal{S}(\mathcal{F})$ and ω satisfies the Properties (a)–(c) in Lemma 7.12. By the uniqueness statement in Lemma 7.12, we obtain $\omega = \omega_{\mathcal{F}} = f(\mathcal{F})$. This shows that the map f is surjective and the proof is complete. $\quad\square$

7.16 Remark In [RS] Ragnarsson and Stancu proved that there is a bijection between the set $\mathrm{Fus}^*(S)$ of saturated fusion systems on S and the set $\mathrm{Idem}^*(S) := \mathbb{Z}_{(p)} B^{\Delta}(S,S) \cap \mathrm{Idem}(S)$. The correspondence associates to $\omega \in \mathrm{Idem}^*(S)$ the fusion system $\mathcal{F}$ satisfying $\mathcal{S}(\mathcal{F}) = \mathrm{Fix}(\omega)$. Thus, the bijection $f\colon \mathrm{Fus}(S) \xrightarrow{\ \sim\ } \mathrm{Idem}(S)$ is an extension of their bijection $\mathrm{Fus}^*(S) \to \mathrm{Idem}^*(S)$. Note that Equation (35) yields a very explicit description of $\omega_{\mathcal{F}}$ in terms of the fixed points of $\omega_{\mathcal{F}}$ that was not apparent in [RS].

We learnt that, in the case where $\mathcal{F}$ is a saturated fusion system, Equation (35) was independently proved by S. Reeh, cf. [Re, Theorem 2.4.11].

If $\mathcal{F}$ is a saturated fusion system then

$$\tilde{\sigma}^{\mathcal{F}}(\omega_{\mathcal{F}}) \in \prod_{P \in \tilde{\mathcal{F}}} \mathrm{End}_{\mathbb{Z}_{(p)} \mathrm{Out}_{\mathcal{F}}(P)}(\mathbb{Z}_{(p)} \overline{\mathrm{Hom}}_{\mathcal{F}}(P, S)) \,.$$

In fact, the $\mathbb{Q}$-algebra isomorphism from Theorem 5.7 (with $R = \mathbb{Q}$) restricts to the injective $\mathbb{Z}_{(p)}$-algebra homomorphism in (22) with $R = \mathbb{Z}_{(p)}$.

The previous remark and our description of $\tilde{\sigma}^{\mathcal{F}}(\omega_{\mathcal{F}})$ in 7.11 lead to the following corollary.

7.17 Corollary *Let $\mathcal{F}$ be a saturated fusion system on S, and let $P \leqslant S$. Then*

$$\frac{|S|}{|\mathrm{Hom}_{\mathcal{F}}(P, S)||C_S(P)|} \in \mathbb{Z}_{(p)}. \tag{37}$$

One might ask whether the converse of Corollary 7.17 is also true, that is, whether every fusion system $\mathcal{F}$ on S satisfying the Condition (37) for every $P \leqslant S$ has to be saturated. However, this is not the case, as the following example shows.

7.18 Example Suppose that $p = 2$ and that $S = \langle x, y \mid x^2 = y^2 = 1, xy = yx \rangle$ is a Klein four-group. Let $\mathcal{F}$ be the fusion system on S, defined in Examples 7.4(c). Hence, $\mathrm{Aut}_{\mathcal{F}}(S) = 1$ and, for every $Q < S$, we have $\mathrm{Hom}_{\mathcal{F}}(Q, S) = \{\alpha|_Q \mid \alpha \in A\}$ where $\mathfrak{A}_3 \cong A \trianglelefteq \mathrm{Aut}(S) \cong \mathfrak{S}_3$. We set $Q_1 := \langle x \rangle$, $Q_2 := \langle y \rangle$, and $Q_3 := \langle xy \rangle$. Then, for $i, j \in \{1, 2, 3\}$, we have $|\mathrm{Hom}_{\mathcal{F}}(Q_j, Q_i)| = 1$ and therefore $|\mathrm{Hom}_{\mathcal{F}}(Q_i, S)| = 3$. Moreover, $\mathrm{Hom}_{\mathcal{F}}(S, S) = \{\mathrm{id}_S\}$, and $\mathrm{Hom}_{\mathcal{F}}(1, 1) = \{\mathrm{id}_1\}$. Thus, for $Q \leqslant S$, we have

$$\frac{|S|}{|\mathrm{Hom}_{\mathcal{F}}(Q, S)||C_S(Q)|} = \begin{cases} 1, & \text{if } Q \in \{1, S\}, \\ \frac{1}{3}, & \text{if } Q \in \{Q_1, Q_2, Q_3\}. \end{cases}$$

Hence $\mathcal{F}$ satisfies (37) in Corollary 7.17, but $\mathcal{F}$ is not saturated, as we have already seen in Examples 7.4(c).

Although fusion systems on S satisfying Condition (37) in Corollary 7.17 need not be saturated, they still do share some properties with saturated fusion systems on S, as the following proposition shows. In the case of saturated fusion systems, see [BCGLO, Proposition 1.16] and [Re, Lemma 1.6.2] for Part (a), and for instance [L, Proposition 2.5] for Part (b).

For every positive natural number n, we denote by n_p the highest p-power dividing n.

7.19 Proposition *Let $\mathcal{F}$ be a fusion system on S satisfying (37) in Corollary 7.17 and let $P \leqslant S$.*

(a) The number $f_{\mathcal{F}}(P)$ of S-conjugacy classes of fully $\mathcal{F}$-normalized subgroups of S that are $\mathcal{F}$-isomorphic to P is not divisible by p.

(b) The subgroup P is fully $\mathcal{F}$-normalized if and only if P is fully $\mathcal{F}$-centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

Proof Let $[P]_{\mathcal{F}}$ denote the set of subgroups of S that are $\mathcal{F}$-isomorphic to P. Moreover, let $\{P_1, \ldots, P_n\}$ be a transversal for the S-conjugacy classes of subgroups in $[P]_{\mathcal{F}}$ and assume that P_1 is fully $\mathcal{F}$-normalized. We consider the function

$$c \colon [P]_{\mathcal{F}} \to \mathbb{N}, \quad Q \mapsto |C_S(Q)| \cdot |\mathrm{Hom}_{\mathcal{F}}(Q, S)|_p \,.$$

40

Note that the positive integer $c(Q)$ is a p-power and that $c(Q) \leqslant |S|$ for all $Q \in [P]_{\mathcal{F}}$, by the condition in (37). Since the number $|\mathrm{Hom}_{\mathcal{F}}(Q, S)|$ is independent of $Q \in [P]_{\mathcal{F}}$, the function c takes its maximum value precisely at the fully $\mathcal{F}$-centralized subgroups in $[P]_{\mathcal{F}}$. Let $Q \in [P]_{\mathcal{F}}$ be arbitrary. Then

$$|\mathrm{Hom}_{\mathcal{F}}(Q, S)| = |\mathrm{Aut}_{\mathcal{F}}(Q)| \cdot |[P]_{\mathcal{F}}| = |\mathrm{Aut}_{\mathcal{F}}(Q) : \mathrm{Aut}_S(Q)| \cdot |N_S(Q) : C_S(Q)| \cdot |[P]_{\mathcal{F}}|$$

and

$$|[P]_{\mathcal{F}}| = \sum_{i=1}^{n} |S : N_S(P_i)| = |S : N_S(P_1)|(f_{\mathcal{F}}(P) + pm_P)$$

for some integer m_P. Therefore,

$$c(Q) = |\mathrm{Aut}_{\mathcal{F}}(Q) : \mathrm{Aut}_S(Q)|_p \cdot \frac{|N_S(Q)|}{|N_S(P_1)|} \cdot |S| \cdot (f_{\mathcal{F}}(P) + pm_P)_p \,. \tag{38}$$

Now assume that Q is fully $\mathcal{F}$-normalized. Then the last equation becomes

$$c(Q) = |\mathrm{Aut}_{\mathcal{F}}(Q) : \mathrm{Aut}_S(Q)|_p \cdot (f_{\mathcal{F}}(P) + pm_P)_p \cdot |S| \,.$$

Since $c(Q) \leqslant |S|$, we obtain $|\mathrm{Aut}_{\mathcal{F}}(Q) : \mathrm{Aut}_S(Q)|_p = 1$, i.e., $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$, and $p \nmid f_{\mathcal{F}}(P)$. Moreover, we obtain $c(Q) = |S|$ and, therefore, that the maximal value of c is equal to $|S|$ and that it is achieved at Q. Thus, Q is fully $\mathcal{F}$-centralized. This shows Part (a) and one direction of Part (b).

Conversely, assume that $Q \in [P]_{\mathcal{F}}$ is fully $\mathcal{F}$-centralized and that $|\mathrm{Aut}_{\mathcal{F}}(Q) : \mathrm{Aut}_S(Q)|_p = 1$. Then $c(Q) = |S|$, the maximal value of c. On the other hand, by Equation (38) and by Part (a), we have

$$c(Q) = \frac{|N_S(Q)|}{|N_S(P_1)|} \cdot |S| \,.$$

This implies $|N_S(Q)| = |N_S(P_1)|$, and Q is fully $\mathcal{F}$-normalized. This completes the proof of Part (b). $\square$

References

[A]   M. ASCHBACHER: Generation of fusion systems of characteristic 2-type. *Invent. Math.* **180** (2010), 225–299.

[AC]   M. ASCHBACHER, A. CHERMAK: A group-theoretic approach to a family of 2-local finite groups constructed by Levi and Oliver. *Ann. of Math. (2)* **171** (2010), 881–978.

[BX]   R. BOLTJE, B. XU: On p-permutation equivalences: between Rickard equivalences and isotypies. *Trans. Amer. Math. Soc.* **360** (2008), 5067–5087.

[Bc96a] S. BOUC: Foncteurs d'ensembles munis d'une double action. *J. Algebra* **183** (1996), 664–736.

[Bc96b] S. BOUC: Construction de foncteurs entre catégories de G-ensembles. *J. Algebra* **183** (1996), 737–825.

[Bc00] S. BOUC: Burnside rings. In: Handbook of algebra, Vol. 2, 739-804, North-Holland, Amsterdam, 2000.

[Bc06]   S. Bouc: The Dade group of a p-group. *Invent. Math.* **164** (2006), 189–231.

[Bc10]   S. Bouc: Biset functors for finite groups. Lecture Notes in Mathematics, 1990. Springer-Verlag, Berlin, 2010.

[BCGLO]   C. Broto, N. Castellana, J. Grodal, R. Levi, B. Oliver: Extensions of p-local finite groups. *Trans. Amer. Math. Soc.* **359** (2007), 3791–3858.

[BLO03]   C. Broto, R. Levi, B. Oliver: The homotopy theory of fusion systems. *J. Amer. Math. Soc.* **16** (2003), 779–856.

[BLO04]   C. Broto, R. Levi, B. Oliver: The theory of p-local groups: a survey. In: Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, 51–84, Contemp. Math., 346, Amer. Math. Soc., Providence, RI, 2004.

[Br]   M. Broué: On Scott modules and p-permutation modules: an approach through the Brauer morphism. *Proc. Amer. Math. Soc.* **93** (1985), 401–408.

[CR]   C. W. Curtis, I. Reiner: Methods of representation theory, Vol. 2. J. Wiley & Sons, New York 1987.

[D1]   A. W. M. Dress: Contributions to the theory of induced representations. In: ed. H. Bass, Algebraic K-Theory II, Lecture Notes in Mathematics 342, Springer-Verlag, Berlin Heidelberg New York 1973, 183–240.

[D2]   A. W. M. Dress: A characterization of solvable groups. *Math. Zeitschr.* **110** (1969), 213–217.

[Ga]   P. Gabriel: Des catégories abéliennes. *Bull. Soc. Math. France* **90** (1962), 323–448.

[Gl]   D. Gluck: Idempotent formula for the Burnside algebra with applications to the p-subgroup simplicial complex. *Illinois J. Math.* **25** (1981), 63–67.

[L]   M. Linckelmann: Introduction to fusion systems. In: Group representation theory, 79–113, EPFL Press, Lausanne, 2007.

[L2]   M. Linckelmann: Trivial source bimodule rings for blocks and p-permutation equivalences. *Trans. Amer. Math. Soc.* **361** (2009), 1279–1316.

[MP]   J. Martino, S. Priddy: Stable homotopy classification of $BG_p^\wedge$. *Topology* **34** (1995), 633–649.

[N-T]   H. Nagao, Y. Tsushima: Representations of finite groups. Academic Press, Boston, 1989.

[R]   K. Ragnarsson: Classifying spectra of saturated fusion systems. *Algebr. Geom. Topol.* **6** (2006), 195–252.

[RS]   K. Ragnarsson, R. Stancu: Saturated fusion systems as idempotents in the double Burnside ring. Preprint. arXiv:0911.0085.

[Re]   S. P. Reeh: Burnside rings of fusion systems. Master's Thesis. University of Copenhagen, 2010.

[Ri]   J. RICKARD: Splendid equivalences: Derived categories and permutation modules. *Proc. London Math. Soc.* **72** (1996), 331–358.

[W]   P. WEBB: Stratifications and Mackey functors II: globally defined Mackey functors. *J. K-Theory* **6** (2010), 99–170.